Supplementary Note for "Asymmetric All-Pay Contests with Heterogeneous Prizes"

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Abstract

This note contains the results in linear algebra that are important to "Asymmetric All-Pay Contests with Heterogeneous Prizes". There are two main results, Lemma 11 and 14, which are used to prove Propositions 4 and 5 in the paper.

We follow the same notations as in "Asymmetric All-Pay Contests with Heterogeneous Prizes", which is referred as to the "main paper" hereafter.

Define $\mathbf{D}_j \equiv \mathbf{1}_j\mathbf{1}_j' - \mathbf{I}_j$, where $\mathbf{1}_j$ is a j-dimensional vector of ones. The diagonal entries of $\mathbf{D}_j$ are zeros, and all the other entries are 1. $\mathbf{B}_j$ is $\mathbf{D}_j$ with the entry at position $(1,1)$ replaced with 1.

Lemma 1 $\det \mathbf{D}_j = (j-1)(-1)^{j-1}$.

Proof. We use induction in this proof. When $j=3$, it is easy to see that

$$\det \mathbf{D}_j = (j-1)(-1)^{j-1} \quad (1)$$

$$\det \mathbf{B}_j = (-1)^{j-1} \quad (2)$$

Suppose the two equations above are true for $j-1$. Expand $\det \mathbf{D}_j$ according the first column, we get a sum of $j-1$ terms of alternating signs. For the j_1th term, put its j_1th column to left and move columns 1 to j_1-1 one position to the right. Then, each term is $-\det \mathbf{B}_{j-1}$, and we have

$$\det \mathbf{D}_j = -(j-1)\det \mathbf{B}_{j-1} \quad (3)$$

Expand $\det \mathbf{B}_j$ according to the first column, we get a sum of j terms of alternating signs. For the (j_1+1)th term and $1 \le j_1 \le j-1$, put its j_1th column to left and move columns 1 to j_1-1 one position to the right. Then, each of the last $j-1$ term is $-\det \mathbf{B}_{j-1}$, so

$$\begin{aligned}\det \mathbf{B}_j &= \det \mathbf{D}_{j-1} - (j-1)\det \mathbf{B}_{j-1} \\ &= \det \mathbf{D}_{j-1} - \det \mathbf{D}_j \end{aligned} \quad (4)$$

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where the second equality comes from (3). Therefore, (3) and (4) imply (1) and (2) are also true for j. ■

If a non-zero entry of $\mathbf{D}_j$ is replace with 0, then we say that the resulting matrix has an *off-diagonal zero* at position (j_1, j_2).

Denote $\mathcal{M}_j$ as the set of all $j \times j$ matrices such that i) it has at most j off-diagonal zeros, ii) each column has at most one off-diagonal zero.

Lemma 2 *If i) $\mathbf{A}_j \in \mathcal{M}_j$, ii) it has j off-diagonal zeros and iii) each row has an off-diagonal zero,* $\det \mathbf{A}_j$ *is* 0 *or has sign* $(-1)^{j-1}$.

Proof. Suppose the off-diagonal zero in row 1 is at column j_2, where $j_2 \neq 1$. Add all other rows to row 1 and divide it by $j-2$, then we get a row of ones. It easy to see that column j_2 does not have an off-diagonal zero. Suppose the off-diagonal zero in row j_2 is at column j_3.

Deduct column j_2 from column j_3, then column j_3 becomes zeros except -1 in row j_3.

Expand the determinant according to column j_3, we get $-\det \mathbf{A}^1_{j-1}$ where $\mathbf{A}^1_{j-1}$ is a $(j-1) \times (j-1)$ matrix with ones in the first row[1]. Moreover, two column of $\mathbf{A}^1_{j-1}$ has no off-diagonal zeros and any other column has one off-diagonal zero. Suppose the two columns without off-diagonal zeros are column j'_1 and j'_2.

Multiply row 1 by $j-3$ and deduct all others rows from it, then the first row has two off-diagonal zeros at column j'_1 and j'_2. Hence the resulting matrix is in $\mathcal{M}_{j-1}$, so its determinant is either 0 or of the sign $(-1)^{j-2}$. As a result, $-\det \mathbf{A}^1_{j-1}$ is 0 or has sign $(-1)^{j-1}$. ■

Lemma 3 *If i) $\mathbf{A}_j \in \mathcal{M}_j$, ii) it has j off-diagonal zeros, iii) at least one row has no off-diagonal zero,* $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$.

Proof. Denote the row without an off-diagonal zero as row j_1. Suppose row j_2 is a row with an off-diagonal zero. Add all the other rows to row j_2, then divide it by $j-1$, then row j_2 only has ones.

Deduct row j_1 from row j_2, we get a row of zeros except 1 at column j_1.

Expand the determinant according to row j_2, we have $(-1)^{j_1+j_2} \det \mathbf{A}^1_{j-1}$ where $\mathbf{A}^1_{j-1}$ is the (j_2, j_1) minor matrix of $\mathbf{A}_j$.

It is easy to see that $j_1 \neq j_2$. Suppose $j_1 > j_2$. Move column j_2 of $\mathbf{A}_{j-1}$ to the left and shift the column 1 to $j_2 - 1$ to the right by one position. We have $(-1)^{j_1+j_2} (-1)^{j_2-1} \det \mathbf{A}^2_{j-1}$. Move row $j_1 - 1$ to the top and shift all the rows above row $j_1 - 1$ down by one position, we have $(-1)^{j_1+j_2} (-1)^{j_2-1} (-1)^{j_1-2} \det \mathbf{A}^3_{j-1} = -\det \mathbf{A}^3_{j-1}$. The first row of $\mathbf{A}^3_{j-1}$ only has ones, and each column has at most one off-diagonal zero. If $j_1 < j_2$, we get the same result similarly.

Multiply row 1 of $\mathbf{A}^3_{j-1}$ by $j-3$ and deduct all the other rows from it, the resulting matrix has one off-diagonal zero in each column. Therefore, this matrix is in $\mathcal{M}_{j-1}$, and has a determinant that is either 0 or of the sign $(-1)^{j-2}$. Since $\det \mathbf{A}_j$ has the opposite sign of the determinant of the resulting matrix, $\det \mathbf{A}_j$ is 0 or has the sign $(-1)^{j-1}$. ■

[1] The rest of the notes use elementary operation of matrix, and we use supscripts to index the matrices in a sequence of such operations.

Off-Diagonal Condition: Column j_1 has an off-diagonal zero if there is a column with an off-diagonal zero in row j_1.

Lemma 4 *If i)* $\mathbf{A}_j \in \mathcal{M}_j$*, ii) it has less than* j *off-diagonal zeros, iii) it does not satisfy the off-diagonal condition,* $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$.

Proof. Since $\mathbf{A}_j$ does not satisfy the off-diagonal condition, there is a column, j_2, such that i) column j_2 has an off-diagonal zero in row j_1, ii) column j_1 has no off-diagonal zero.

Deduct column j_2 from column j_1, and and then column j_1 becomes zeros except a 1 in row j_2. The following analysis is similar to Lemma 2. Expand the determinant according to column j_1, and then we have $(-1)^{j_1+j_2} \det \mathbf{A}^1_{j-1}$, where $\mathbf{A}^1_{j-1}$ is the (j_2, j_1) minor matrix of $\mathbf{A}_j$. Recall that $j_2 \neq j_1$, so first consider $j_2 < j_1$. Move row $j_1 - 1$ to the top and column $j_2 - 1$ to the right. The determinant becomes $-\det \mathbf{A}^2_{j-1}$. Note that the entry at $(1, 1)$ in $\mathbf{A}^2_{j-1}$ is the entry at (j_1, j_2) in $\mathbf{A}_j$, which is 0 by assumption. Therefore, $\mathbf{A}^2_{j-1} \in \mathcal{A}_{j-1}$, so $\det \mathbf{A}_j$ is either 0 or of the sign $(-1)^{j-1}$. If $j_1 > j_2$, we can get the same result similarly. ■

Lemma 5 *If i)* $\mathbf{A}_j \in \mathcal{M}_j$*, ii) it has less than* j *off-diagonal zeros, iii) it satisfies the off-diagonal condition,* $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$.

Proof. First, we claim that column j_1 has an off-diagonal zero if row j_1 has an off-diagonal zero. To see why, suppose otherwise. Then, row j_1 has an off-diagonal zero at column j_2 and column j_1 does not. Column j has an off-diagonal zero in row j_1, and column j_1 has no off-diagonal zero, which contradicts the off-diagonal condition.

As a result, if column j_1 has no off-diagonal zero, row j_1 has no off-diagonal zero. Denote $\mathcal{J}$ as $\{1, 2, ..., j\}$ and $\mathcal{H}$ as the columns with an off-diagonal zero, then $\mathcal{J} \backslash \mathcal{H}$ is a set of rows without off-diagonal zeros.[2]

Pick any row with an off-diagonal zero and add all the other rows with off-diagonal zeros to it. The resulting row is either $\hat{\jmath}$ or $\hat{\jmath} - 1$, where $\hat{\jmath}$ is the number of rows with off-diagonal zeros. Moreover, in this row, $\hat{\jmath} - 1$ is at the columns in $\mathcal{H}$ and $\hat{\jmath}$ is at the columns in $\mathcal{J} \backslash \mathcal{H}$.

Pick a row in $\mathcal{J} \backslash \mathcal{H}$ and add the rest to this row, the resulting row has entries equal $\bar{\jmath}$ or $\bar{\jmath} - 1$, where $\bar{\jmath} = |\mathcal{J} \backslash \mathcal{H}|$. Moreover, $\bar{\jmath}$ is in the columns in $\mathcal{H}$, and $\bar{\jmath} - 1$ is in the columns in $\mathcal{J} \backslash \mathcal{H}$.

Add the row with $\hat{\jmath}$ and $\hat{\jmath} - 1$ to the one with $\bar{\jmath}$ and $\bar{\jmath} - 1$, and divide it by $\bar{\jmath} + \hat{\jmath} - 1$. The resulting row has only ones.

This row of ones replaces a row with one off-diagonal zero. Deduct a row without an off-diagonal zero from this row of ones, then we get a row of zeros except one entry as 1. Similar to Lemmas 3 and 4, if we expand the determinant according to this row and move some rows and columns, $\det \mathbf{A}_j$ becomes $-\det \mathbf{A}_{j-1}$, where $\mathbf{A}_{j-1} \in \mathcal{M}_{j-1}$. Hence, $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$. ■

Lemma 6 *If* $\mathbf{A}_j \in \mathcal{M}_j$, $\det \mathbf{A}_j = 0$ *or* $\det \mathbf{A}$ *has the sign* $(-1)^{j-1}$.

Proof. By induction.

It is easy to verify that the statement is true for $j' = 2$. Suppose the statement is true for $j' = j - 1$, Lemma 2 to 5 show that it is also true for $j' = j$. Therefore the Lemma is true if integer j is bigger than 1. ■

[2] There might be more than one such set.

Lemma 7 $\det \mathbf{H}_j$ *has sign* $(-1)^{j-1}$, *where* $\mathbf{H}_j$ *is a* $j \times j$ *matrix with zero diagonal entries and* $h_{j_1,j_2} = \sum_{l=1}^{j} h_l - h_{j_1} - h_{j_2}$, *where* $h_l > 0$ *for any* l.

Proof. Column 1 of $\mathbf{H}_j$ is a sum of $j-1$ vectors, $\sum_{l=2}^{j} h_l \mathbf{1}_{-1,-l}$, where $\mathbf{1}_{-j_1,-j_2}$ is a column vector with ones except two zeros in row j_1 and j_2. Therefore, $\det \mathbf{H}_j = \sum_{l=2}^{j} \det \mathbf{H}_j^1$, where $\mathbf{H}_j^1$ is a $j \times j$ matrix with column 1 as $h_l \mathbf{1}_{-1,-l}$ and the other columns the same as in $\mathbf{H}_j$. Note that column 1 in $\mathbf{H}_l^1$ only contains 0 or h_l.

For any $\mathbf{H}_l^1$, its second column is $\sum_{l=1,3,\ldots,j} h_l \mathbf{1}_{-2,-l}$, so $\det \mathbf{H}_l^1$ also equals a sum of $j-1$ determinants of $j \times j$ matrices. Moreover, the first two columns of these matrices only have one h_l.

Repeat this step for the other columns until $\det \mathbf{H}_j$ become a sum of determinants of $j \times j$ matrices that have zero or the same h_l in each column. Moreover, these determinants have other properties. First, if column j_2 of these matrices has only h_l, then it is $h_l \mathbf{1}_{-l,-j_2}$. Second, column j_2 cannot have h_{j_2}.

Denote $\mathcal{J} = \{1, ..., j\}$ and

$$\mathfrak{K}_j = \left\{ (\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j) \mid \mathcal{J}_l \cap \mathcal{J}_{l'} = \varnothing,\ \cup_{l=1}^{j} \mathcal{J}_l = \mathcal{J},\ \text{and } |\mathcal{J}_l| < j \right\}$$

$(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j)$ is a j-set partition of $\mathcal{J}$ except that $\mathcal{J}_l$ can be empty. For any $(\mathcal{J}_1, ..., \mathcal{J}_j) \in \mathfrak{K}_j$, replace the entry of (j_1, j_2) in $\mathbf{D}_j$ with 0 if $j_2 \in \mathcal{J}_{j_1}$, and denote the resulting matrix as $\mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$. $\det \mathbf{H}_j$ is a polynomial of order j, and each term has the same order. That is

$$\det \mathbf{H}_j = \sum_{(\gamma_1,\ldots,\gamma_j)} \left(\eta_{\gamma_1,\ldots,\gamma_j} \prod_{l=1}^{j} h_l^{\gamma_l} \right)$$

where the sum is over the set $\left\{ (\gamma_1, ..., \gamma_j) \mid \gamma_l < j,\ \sum_{l=1}^{j} \gamma_l = j \text{ and } \gamma_l \in \mathbb{Z}_+ \right\}$. Denote $\mathfrak{K}_{\gamma_1,\ldots,\gamma_j} = \{(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j) \in \mathfrak{K}_j \text{ such that } |\mathcal{J}_l| = \gamma_l\}$. Then, $\eta_{\gamma_1,\ldots,\gamma_j} = \sum_{(\mathcal{J}_1,\ldots,\mathcal{J}_j)} \det \mathbf{A}_{\mathcal{J}_1,\ldots,\mathcal{J}_j}$, where the sum is over the set $\mathfrak{K}_{\gamma_1,\ldots,\gamma_j}$.

For each $(\mathcal{J}_1, ..., \mathcal{J}_j)$ in $\mathfrak{K}_{\gamma_1,\ldots,\gamma_j}$, $\mathbf{A}_{\mathcal{J}_1,\ldots,\mathcal{J}_j}$ is in $\mathcal{A}_j$, so $\eta_{\gamma_1,\ldots,\gamma_j}$ is either 0 or has sign $(-1)^{j-1}$ by Lemma 6. As a result, $\det \mathbf{H}_j$ either is 0 or has sign $(-1)^{j-1}$. Now we are going to show $\det \mathbf{H}_j \neq 0$ by proving that one of the coefficients is not zero.

Consider the coefficient of $h_1^{j-2} h_2 h_3$ in $\det \mathbf{H}_j$, it is a sum of determinants and one of them is associated with $(\mathcal{J}_1, ..., \mathcal{J}_j)$ such that $\mathcal{J}_2 = \{3\}, \mathcal{J}_3 = \{1\}$ and $\mathcal{J}_1 = \mathcal{J} \setminus (\mathcal{J}_2 \cup \mathcal{J}_3)$. Such determinant has j off-diagonal zeros: one at column 3 to row 2, another at column 1 row 3 and all the others are in the first row. The resulting matrix has zeros in the first row except a 1 in column 3. Expand this determinant according to the first row, we get a $(j-1) \times (j-1)$ determinant. Switch the first two row and we get $-\det \mathbf{D}_{j-1}$ which is not zero. ■

Lemma 8 *$\det\hat{\mathbf{H}}_j$ has sign $(-1)^{j-1}$, where $\hat{\mathbf{H}}_j$ is a $j\times j$ matrix with zeros diagonal entries and $\hat{h}_{j_1,j_2}=\sum_{l=1}^{j'}h_l-h_{j_1}-h_{j_2}$, where $h_l>0$ for any l and $j'\geq j$.*

Proof. Add $\sum_{l=j}^{j'}h_l$ to the entries off the diagonal of $\mathbf{H}_j$, we have $\hat{\mathbf{H}}_j$. Denote $h_l'=h_l+\frac{1}{j-2}\sum_{l=j}^{j'}h_l>0$, then $\hat{h}_{j_1,j_2}=\sum_{l=1}^{j}h_l'-h_{j_1}'-h_{j_2}'$ and the previous lemma implies that $\det\hat{\mathbf{H}}_j$ *has sign* $(-1)^{j-1}$. ■

Let us introduce some notations before we move to the next lemma. Suppose the active and participating players remain the same for a given interval. Let $\mathcal{A}$ be the active players and $\mathcal{P}$ be the participating players, and $\mathcal{A}\subset\mathcal{P}$. Then, for any s in this interval, equilibrium strategies $(G_i)_{i\in\mathcal{A}}$ solve

$$W\left(\mathbf{G}_{-i}(s),\mathbf{v}\right)-C_i(s)=u_i \qquad \text{for } i\in\mathcal{A} \tag{5}$$

where $\mathbf{G}_{-i}(s)=(G_i(s))_{i\in\mathcal{P}\setminus\{i\}}$ and $\mathbf{v}=(v^k)_{k\in\mathcal{P}}$.[3] Note that if we omit the superscripts in (35) of the main paper, the equation is the same as (5).

Proposition 1 implies that $\mathcal{A}$ and $\mathcal{P}$ contain consecutive players, that is, $i\in\mathcal{A}$ if $i-1$ and $i+1\in\mathcal{A}$. For simpler notation, suppose $\mathcal{A}=\mathcal{P}=\{1,2,...,j'\}$. If the strongest player in $\mathcal{P}$ is not 1, or if $\mathcal{A}\subsetneq\mathcal{P}$, the analysis is the same.

For any $i'\in\mathcal{A}\setminus\{i\}$, take derivatives of both sides of (5) with respect to $G_{i'}$, we have

$$\sum_{l\in\mathcal{A}\setminus\{i,i'\}}\frac{\partial W\left(\mathbf{G}_{-i}(s),\mathbf{v}\right)}{\partial G_l(s)}\frac{\partial G_l(s)}{\partial G_{i'}(s)}=-\frac{\partial W\left(\mathbf{G}_{-i}(s),\mathbf{v}\right)}{\partial G_{i'}(s)} \quad \text{for } i\neq i' \tag{6}$$

$$\sum_{l\in\mathcal{A}\setminus\{i'\}}\frac{\partial W\left(\mathbf{G}_{-i'}(s),\mathbf{v}\right)}{\partial G_l(s)}\frac{\partial G_l(s)}{\partial G_{i'}(s)}=0 \tag{7}$$

We can write (6) and (7) into matrix form

$$\begin{aligned}\hat{\mathbf{J}}_{j-1}\boldsymbol{\delta} &= -\mathbf{d}\\ \mathbf{d}'\boldsymbol{\delta} &= \mathbf{0}\end{aligned} \tag{8}$$

where $j=|\mathcal{A}|$ and $\hat{\mathbf{J}}_{j-1}$ is a $(j-1)\times(j-1)$ matrix, $\boldsymbol{\delta}$ and $\mathbf{d}$ are vectors of $j-1$ rows. The diagonal entries of $\hat{\mathbf{J}}$ are zero and the entry at (j_1,j_2) is $\partial W\left(\mathbf{G}_{-j_1}(s),\mathbf{v}\right)/\partial G_{j_2}(s)$; the element in row j_1 of $\boldsymbol{\delta}$ is $\partial G_{j_1}(s)/\partial G_{i'}(s)$ for $j_1\neq i'$;the element in row j_1 of $\mathbf{d}$ is $\partial W\left(\mathbf{G}_{-j_1}(s),\mathbf{v}\right)/\partial G_{i'}(s)$ for $j_1\neq i'$. Define $\mathbf{J}_j=\begin{pmatrix}\hat{\mathbf{J}}_{j-1} & \mathbf{d}\\ \mathbf{d}' & 0\end{pmatrix}$ and w_{j_1,j_2} as the entry in row j_1 and column j_2 of $\mathbf{J}_j$.

Define first order difference as $\Delta_k^{(1)}=v^{k-1}-v^k$, for $k=2,...,j'$. The lth order difference is $\Delta_k^{(l)}=\Delta_{k-1}^{(l-1)}-\Delta_k^{(l-1)}$ for $l=1,...,j'-1$, and $k=l+1,...,j'$.

[3] Since the indices of $\mathbf{G}$ and $\mathbf{v}$ in $W\left(\mathbf{G}_{-i},\mathbf{v}\right)$ are the same below, we sometime only mention the indices for $\mathbf{G}$.

Lemma 9

$$w_{j_1,j_2} = \Delta_j^{(1)} + \sum_{l'=2}^{j'-1} \Delta_{j'}^{(l')} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma_{j_1,j_2}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right)$$

where $j_1 \neq j_2$ *and* $\Gamma_{j_1,j_2} = \{1, 2, ..., j'\} \backslash \{j_1, j_2\}$.

Proof. We are going to prove by induction. First, it is easy to verify the statement is true for $j' = 3$. Suppose the statement is true for $j' = j'_I - 1$, we are going to show that it is also true for $j' = j'_I$.

For the purpose of cleaner exhibition, the following proof only focuses on w_{12}.

We know that

$$\begin{aligned}
w_{12} &= W\left(G_3, ..., G_{j'_I}, \Delta_2^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\
&= G_{j'_I} W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)}, ..., \Delta_{j'_I-1}^{(1)}\right) + \left(1 - G_{j'_I}\right) W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\
&= W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\
&\quad + G_{j'_I}\left(W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)}, ..., \Delta_{j'_I-1}^{(1)}\right) - W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\right) \\
&= W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\
&\quad + G_{j'_I} W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)} - \Delta_3^{(1)}, \Delta_3^{(1)} - \Delta_4^{(1)}, ..., \Delta_{j'_I-1}^{(1)} - \Delta_{j'_I}^{(1)}\right) \\
&= W\left(G_3, ..., G_{k'_I-1}, \Delta_3^{(1)}, ..., \Delta_{k'_I}^{(1)}\right) + G_{k'_I} W\left(G_3, ..., G_{k'_I-1}, \Delta_3^{(2)}, ..., \Delta_{k'_I}^{(2)}\right) \qquad (9)
\end{aligned}$$

Since the statement is true for $k' = j'_I - 1$, we have

$$W(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}) = \Delta_{j'_I}^{(1)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l')} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \qquad (10)$$

$$W(G_3, ..., G_{j'_I-1}, \Delta_3^{(2)}, , ..., \Delta_{j'_I}^{(2)}) = \Delta_{j'_I}^{(2)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l'+1)} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \qquad (11)$$

where $\Gamma'_{12} = \{1, 2, ..., j'_I - 1\} \backslash \{1, 2\}$.

Substitute (10) and (11) into (9), then we have

$$\begin{aligned}
w_{12} &= \Delta_{j'_I}^{(1)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l')} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \\
&\quad + G_{j'_I} \left(\Delta_{j'_I}^{(2)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l'+1)} \left(\sum_{\{i_1,...,i_{l'-1}\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \right)
\end{aligned}$$

therefore the coefficient of $\Delta_{j'_I}^{(j')}$ is

$$\left(\sum_{\{i_1,\ldots,i_{l'-1}\}\subset\Gamma'_{12}}\left(\prod_{l=1}^{l'-1}G_{i_l}\right)\right)+G_{j'_I}\left(\sum_{\{i_1,\ldots,i_{l'-2}\}\subset\Gamma'_{12}}\left(\prod_{l=1}^{l'-2}G_{i_l}\right)\right)$$
$$=\left(\sum_{\{i_1,\ldots,i_{l'-1}\}\subset\Gamma'_{12}\cup\{G_{j'_I}\}}\left(\prod_{l=1}^{l'-1}G_{i_l}\right)\right)$$

As a result, $w_{12}=\Delta_{j'}^{(1)}+\sum_{l'=2}^{j'-1}\Delta_{j'}^{(l')}\left(\sum_{\{i_1,\ldots,i_{l'-1}\}\subset\Gamma_{12}}\left(\prod_{l=1}^{l'-1}G_{i_l}\right)\right)$.

Similarly, we can extend the analysis above to w_{j_1,j_2} for $j_1\neq j_2$. Hence, the statement is also true for j'_I. ■

Under the assumption of QPS, $\Delta_{j'}^{(l)}=0$ for $l>2$. Therefore, both $\mathbf{J}_j$ and $\hat{\mathbf{J}}_j$ are simplified, and

$$\begin{aligned}w_{j_1,j_2} &= \left(\sum_{l=1}^{j'}G_l-G_{j_1}-G_{j_2}\right)\left(v^{j'-2}-2v^{j'-1}\right)+\left(v^{j'-1}-v^{j'}\right)\\ &= \left(\sum_{l=1}^{j'}G_l-G_{j_1}-G_{j_2}\right)\Delta_{j'}^{(2)}+\Delta_{j'}^{(1)}\end{aligned}$$

if $j_1\neq j_2$.

If $\mathcal{A}^*(s)=\mathcal{P}^*(s)=\{1,2,...,j\}$, matrix $\hat{\mathbf{J}}_j$ is exactly $\mathbf{J}_{\mathcal{A}^*(s)}$ in the proof of Proposition 3 in the main paper. Lemmas 10 and 13 determine the sign of $\mathbf{J}_{\mathcal{A}^*(s)}$.

Lemma 10 $\det\mathbf{J}_j$ *and* $\det\hat{\mathbf{J}}_j$ *have sign* $(-1)^{j-1}$ *if the prizes satisfy QPS and* $G_l>0$ *for* $l\in\mathcal{P}$.[4]

Proof. First, suppose $\mathcal{A}=\mathcal{P}$, so $j=j'$.

$\det\mathbf{J}_j=\left(\Delta_{j'}^{(1)}\right)^j\det\mathbf{Z}_j$, where $z_{j_1,j_2}=\sum_{l=1}^{j}h_l-h_{j_1}-h_{j_2}, h_l=G_l\Delta_{j'}^{(2)}/\Delta_{j'}^{(1)}+\frac{1}{j-2}$.

Assume any $G_{i'}$ equals G_i for $i'\in\mathcal{P}\backslash\{i,j\}$, we have $\partial W_i/\partial G_j=(j-2)\,\Delta_{j'}^{(2)}G_i+\Delta_{j'}^{(1)}>0$, where the inequality comes from Lemma 1 in the main paper. As a result, $h_i=G_i\Delta_{j'}^{(2)}/\Delta_{j'}^{(1)}+\frac{1}{j-2}>0$.

Lemma 7 implies $\det\mathbf{Z}_j$ is of the sign $(-1)^{j-1}$, and so it is $\det\mathbf{J}_j$.

Second, suppose $\mathcal{A}\subsetneq\mathcal{P}$, then we have $j<j'$.

$$w_{j_1,j_2}=\left[\Delta_{j'}^{(2)}\left(\sum_{l\in\mathcal{A}}G_l-G_{j_1}-G_{j_2}\right)+\Delta_{j'}^{(1)}\right]+\Delta_{j'}^{(2)}\sum_{l\in\mathcal{P}\backslash\mathcal{A}}G_l$$

if $j_1\neq j_2$ and $j_1,j_2\in\mathcal{A}$.

[4] This claim may fail if the prizes are not QPS or GPS. Consider a four-player contest with prizes $v_1=7,v_2=2,v_3=1$ and $v_4=0$. If G_1 and G_2 are close to 0, G_3 and G_4 are close to 1, $\det\mathbf{J}_4$ is close to 5.

$\det \mathbf{J}_j = (\Delta_{j'}^{(1)})^j \det \mathbf{Z}_j$, where $z_{j_1,j_2} = \sum_{l \in \mathcal{A}} h_l - h_{j_1} - h_{j_2}$, $h_l = G_l \Delta_{j'}^{(2)} / \Delta_{j'}^{(1)} + \frac{y}{j-2}$, $y = 1 + \left(\Delta_{j'}^{(2)} / \Delta_{j'}^{(1)}\right) \sum_{l \in \mathcal{P} \setminus \mathcal{A}} G_l$. $\det \mathbf{Z}_j$ has sign $(-1)^{j-1}$ according to Lemma 7.

Let us consider $\det \hat{\mathbf{J}}_j$. Suppose $\mathcal{A}' = \mathcal{A} \setminus \{i'\}$ and consider equation (5) with $\mathcal{A}'$ and $\mathcal{P}$. The corresponding $\mathbf{J}_{j-1}$ of the new system is just $\hat{\mathbf{J}}_{j-1}$ in the original system. Therefore, $\det \hat{\mathbf{J}}_{j-1}$ has sign $(-1)^{j-2}$ according to Lemma 7. ■

Lemma 11 *Suppose the prizes satisfies QPS. For any $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$ and $i \in \mathcal{A}$, LHS of (5) for i decreases if $G_i(s)$ increases in other equations of (5).*

Proof. Suppose j is the weakest player in $\mathcal{A}$. Lemma 10 shows that $\det \mathbf{J}_j$ has sign $(-1)^{j-1}$ and $\det \hat{\mathbf{J}}_{j-1}$ has sign $(-1)^{j-2}$, then $\hat{\mathbf{J}}_{j-1}$ is invertible and

$$\mathbf{d}'\boldsymbol{\delta} = -\mathbf{d}'\hat{\mathbf{J}}_{j-1}^{-1}\mathbf{d} = \det \mathbf{J}_j / \det \hat{\mathbf{J}}_{j-1} < 0$$

Therefore, we have $\hat{\mathbf{J}}_{j-1}\boldsymbol{\delta} = -\mathbf{d}$ and $\mathbf{d}'\boldsymbol{\delta} > \mathbf{0}$.

If we take the derivative of LHS of (5) w.r.t. $G_j(s)$, the derivatives for equations $i \neq j$ are $\hat{\mathbf{J}}_{j-1}\boldsymbol{\delta} + \mathbf{d}$ and the derivative for equation $i = j$ is $\mathbf{d}'\boldsymbol{\delta}$. Therefore, $\hat{\mathbf{J}}_{j-1}\boldsymbol{\delta} + \mathbf{d} = \mathbf{0}$ and $\mathbf{d}'\boldsymbol{\delta} > \mathbf{0}$ imply that LHS of (5) for i decreases if $G_j(s)$ increases in other equations of (5), and the lemma is true for player j.

Since players in $\mathcal{A}$ are symmetric in this problem, the lemma is also true for other players in $\mathcal{A}$. ■

Now consider geometric prize sequences.

Lemma 12 $\det \mathbf{H}_j$ *has sign $(-1)^{j-1}$, where $\mathbf{H}_j$ is a $j \times j$ matrix with zeros diagonal entries and $h_{j_1,j_2} = \left(\prod_{l=1}^{j'} h_l\right) / (h_{j_1} h_{j_2})$ with $h_l > 0$ for any l and $j' \geq j$.*

Proof. Multiply row $j_1 > 1$ by h_{j_1}.

Divided column j_2 by $\left(\prod_{l=1}^{j'} h_l\right) / h_{j_2}$. Let us describe the resulting matrix. First, the entries in the first row are $1/h_1$ except a zero at the first column. Second, the diagonal entries are zero. Third, $h_{j_1,j_2} = 1$ for $j_1 > 1$ and $j_1 \neq j_2$.

Multiply the first row by h_1, we get $\det \mathbf{D}_j = (j-1)(-1)^{j-1}$ by Lemma 1. ■

Lemma 13 $\det \mathbf{J}_j$ *and* $\det \hat{\mathbf{J}}_j$ *have sign $(-1)^{j-1}$ if the prizes satisfies GPS and $G_l > 0$ for $l \in \mathcal{P}$.*

Proof. We can verify that $\Delta_{j'}^{(l)} = (\alpha - 1)^l v^j$, so $w_{j_1,j_2} = \prod_{l \in \mathcal{P} \setminus \{j_1,j_2\}} ((\alpha - 1) G_l + 1)$. Denote $h_l = (a-1) G_l + 1$, and $h_l > 0$ since $\alpha > 1$ and $0 < G_l$. Therefore, Lemma 12 implies $\det \mathbf{J}_j$ has sign of $(-1)^{j-1}$.

Similar to the case of QPS, $\det \hat{\mathbf{J}}_{j-1} = \det \mathbf{J}_{j-1}$ for $\mathcal{A}' = \mathcal{A} \setminus \{i'\}$ and $\mathcal{P}$, so $\det \hat{\mathbf{J}}_{j-1}$ has sign $(-1)^{j-2}$. ■

Lemma 14 *Suppose the prizes satisfies GPS, then for any subset $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$ and $i \in \mathcal{A}$, LHS of (5) for i decreases if $G_i(s)$ increases in other equations of (5).*

Proof. Given the previous lemma, the proof is the same as Lemma 11. ∎

The following two lemmas provide the omitted proofs in Lemma 5 in the main paper. Denote $h_l \equiv (\alpha - 1)\, G_l + 1$ and a $j \times j$ matrix $\mathbf{H}_j \equiv \mathbf{J}_j$. Switch the last two columns of $\mathbf{H}_j$, then drop the last column and last row, we have a $(j-1) \times (j-1)$ matrix**,** denote it as $\tilde{\mathbf{J}}_{j-1}$.

Lemma 15 *If the prize sequence is geometric,* $\det \tilde{\mathbf{J}}_{j-1}$ *has sign of* $(-1)^j$.

Proof. Now we are going to use induction to show that $\det \tilde{\mathbf{J}}_{j-1}$ has sign of $(-1)^j$.

First, when $j = 3$, we have $\det \tilde{\mathbf{J}}_2 = \det \begin{pmatrix} 0 & h_2 \\ h_3 & h_1 \end{pmatrix} < 0$.

Suppose $\det \tilde{\mathbf{J}}_{j'-1}$ has sign of $(-1)^{j'}$. Consider $\det \tilde{\mathbf{J}}_{j'}$. First, divide all columns except the last one by $h_{j'+1}$, then times column $j'-1$ by $h_{j'-1}$ and deduct it from the last column. The last column has zeros except in row $j'-1$. Expand the determinant according to the last column, and we have (-1) times a $(j'-1)$-dimensional determinant. Take the transpose of the matrix, we get $\det \tilde{\mathbf{J}}_{j'-1}$ which has the sign of $(-1)^{j'}$. As a result, $\det \tilde{\mathbf{J}}_{j'}$ has sign of $(-1)^{j'-1}$. ∎

Take matrix $\hat{\mathbf{J}}_{j-1}$, replace its ith column by $\mathbf{d}$, and denote it as $\tilde{\mathbf{J}}_{j-1,i}$. As a result, $\tilde{\mathbf{J}}_{j-1,i}$ is a $(j-1) \times (j-1)$ matrix.

Lemma 16 *If the prize sequence is quadratic, we have* $\det \tilde{\mathbf{J}}_{j-1,i} - \det \tilde{\mathbf{J}}_{j-1,i-1}$ *has sign of* $(-1)^j$ *for* $i = 2, ..., j-1$.

Proof. If the prize sequence is quadratic, the H_j has entry $h_{i,i'} = \sum_{l=1}^{j} h_l - h_i - h_{i'}$ for $i \neq i'$ and zero diagonal elements.

We are going to show that $\det \tilde{\mathbf{J}}_i - \det \tilde{\mathbf{J}}_{i-1}$ has sign of $(-1)^j$ after a series of elementary operations. If we compare $\tilde{\mathbf{J}}_i$ and $\tilde{\mathbf{J}}_{i-1}$, they are the same except the $(i-1)$th and ith columns. The $(i-1)$th and ith columns in $\tilde{\mathbf{J}}_i$ are $\mathbf{h}_{i-1}$ and $\mathbf{d}$, and the $(i-1)$th and ith columns in $\tilde{\mathbf{J}}_{i-1}$ are $\mathbf{d}$ and $\mathbf{h}_i$. Therefore

$$
\begin{aligned}
& \det \tilde{\mathbf{J}}_i - \det \tilde{\mathbf{J}}_{i-1} \\
= \;& \det(\mathbf{h}_1, ..., \mathbf{h}_{i-1}, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) - \det(\mathbf{h}_1, ..., \mathbf{d}, \mathbf{h}_i, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) \\
= \;& \det(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_{i-1}, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) + \det(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_i, \mathbf{d}, \mathbf{h}_{i+1} ..., \mathbf{h}_{j-1}) \\
= \;& \det(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_{i-1} + \mathbf{h}_i, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}) \qquad (12)
\end{aligned}
$$

where the second equality comes from switching the $(i-1)$th and ith columns in $\det \tilde{\mathbf{J}}_{i-1}$.

Deduct ith row from $(i-1)$th row, the $(i-1)$th row in the resulting determinant is $h_i - h_{i-1}$ except 0 in the $(i-1)$th column. Divide the $(i-1)$th row by $h_i - h_{i-1}$. Since $h_i - h_{i-1} > 0$, the resulting determinant has the same sign.

Deduct column i from all other columns except the $(i-1)$th column. The $(i-1)$th row of the resulting determinant has zeros except 1 at column i. The other rows are the same as in (12).

Expand the determinant according to the $(i-1)$th row, the result is $-\det \mathbf{Y}_{j-2}$, where $\mathbf{Y}_{j-2}$ is a $(j-2)$-dimensional determinant. Let us describe $\mathbf{Y}_{j-2}$. The $(i-1)$th column of $\mathbf{Y}_{j-2}$ is $h_{i-1}+h_i$ excluding the $(i-1)$th row; Column $i' > i-1$ of $\mathbf{Y}_{j-2}$ has $h_j - h_{i'}$ except $-\sum_{l=1}^{j-1} h_l + h_{i'}$ in column i'; Column $i' < i-1$ of $\mathbf{Y}_{j-2}$ has $h_j - h_{i'+1}$ except $-\sum_{l=1}^{j-1} h_l + h_{i'+1}$ in column i'.

Add all other columns to column $(i-1)$ of $\mathbf{Y}_{j-2}$, the $(i-1)$th column has only $(j-2)\, h_{j-1}$. Since $h_{j-1} > 0$, we can normalize column $i-1$ to ones without change the sign of the determinant.

Multiply column $i-1$ with $h_j - h_{i'}$ and deduct it from column $i' \neq i-1$. Then the i'th column has only zeros except $-\sum_{l=1}^{j-1} h_l + h_{i'+1} - (h_j - h_{i'}) = h_{i'} - \sum_{l \in \mathcal{A}\setminus\{i'\}} h_l < 0$; row $i-1$ has only zeros except 1 at column $i-1$, therefore we can set column $i-1$ to zeros except in row $i-1$ and not affect the sign of the determinant.

The resulting determinant is a diagonal matrix, therefore the determinant equals

$$-\prod_{i' \in \mathcal{A}\setminus\{i-1,j\}} \left(h_{i'} + \sum_{l \in \mathcal{A}\setminus\{i'\}} h_l \right)$$

which is a product of $1+(j-3)$ negative numbers, therefore $\det \tilde{\mathbf{J}}_i - \det \tilde{\mathbf{J}}_{i-1}$ has sign of $(-1)^j$. ■

Suppose $\mathcal{P} = \{1, 2, ..., j+1\}$ and $l = j+1$. If the prize sequence is geometric, matrix $\tilde{\mathbf{J}}_{\mathcal{P}\setminus\{l\},j}$ in (43) in the main paper is $\tilde{\mathbf{J}}_{j-1}$. Therefore, Lemma 15 implies that $\det \tilde{\mathbf{J}}_{\mathcal{P}\setminus\{l\},j} = (-1)^{|\mathcal{P}|}$. If the prize sequence is quadratic, matrix $\tilde{\mathbf{J}}_{\mathcal{P}\setminus\{l\},j}$ in (43) in the main paper is $\tilde{\mathbf{J}}_{j-1}$. Lemmas 16 implies that $\det \tilde{\mathbf{J}}_{\mathcal{P}\setminus\{l,j\}} = (-1)^{|\mathcal{P}|}$. Recall that $\mathcal{A}$ and $\mathcal{P}$ contain consecutive players, the other possibilities are i) the strongest player in $\mathcal{P}$ is not 1, or ii) $\mathcal{A} \subsetneq \mathcal{P}$. Because the proofs for these cases are similar to Lemmas 15 and 16, they are omitted.