Equilibrium Analysis of the All-Pay Contest with Two Nonidentical Prizes: Complete Results*

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Abstract

This paper studies contests in which three or more players compete for two nonidentical prizes. The players have distinct constant marginal costs of performance, which are commonly known. We show that the contests have a unique Nash equilibrium, and it is in mixed strategies. Moreover, we characterize the equilibrium payoffs and strategies in closed form. We also study how the equilibrium payoffs and strategies vary with the prizes. As an application, we numerically compute the optimal allocation of prizes that maximizes the total expected performance of asymmetric players.

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1 Introduction

Contests with asymmetric players and heterogeneous prizes are predominant. For example, students of various intellectual levels compete for different grades, athletes of different abilities compete for different medals, and employees with different experience compete for different promotion opportunities. If we rank the prizes in a contest from the highest value to the lowest, we obtain a nonincreasing sequence of prize values, to which we refer as the prize sequence. The prize sequences in these contests have different shapes. For instance, in the 2016 U.S. Open tennis tournament, the prize is 3.5 million for the winner, 1.75 million for a runner-up, and 0.875 million for a semifinalist. A prize is roughly half of the value of the next higher prize. In contrast, the prizes in the golf tournaments do not have the same property. For example, in 2016 U.S. Open golf tournament, the prizes are 1.8 million for the champion, 1.1 million for the runner-up, and 0.68 million for the third place.

The shape of the prize sequence is especially important if the players have different abilities. To see why, if the prize sequence is very concave, the difference between higher prizes is small relative to that between lower prizes, which leads to less competition among the players with stronger abilities. In contrast, if the prize sequence is very convex, the difference between lower prizes is small relative to that between higher prizes, which leads to less competition among the players with lower abilities. For contest organizers, it is an important decision to allocate a budget of prize money into prizes. For example, in 2012, the Wimbledon tennis tournament increased the lowest prize by 26%, the second lowest by 15%, and the third lowest by 13%. What is the optimal way to split a fixed budget of prize money into prizes such that the total performance is maximized? Moreover, what are the players' resulting payoffs of the performance maximizing prizes? In order to address these questions, we need to understand how different prize structures affect players with different abilities.

In this paper, we consider a complete-information all-pay contest among players of distinct constant marginal costs and two prizes of distinct values. We measure the concavity/convexity of the prize sequence in the ratio of the difference between the two prizes to the difference between the lower prize and zero. This is the simplest setup to introduce prize sequences of different concavity/convexity. We show that the contest has a unique Nash equilibrium, and it is in mixed strategies. In addition, we provide a closed-form characterization of the equilibrium payoffs and strategies.

This paper's contribution is threefold. First, it shows equilibrium uniqueness. The uniqueness is not obvious because multiple equilibria have been found in contests with identical players (e.g. Baye et al. (1996)). In contrast, Siegel (2010) constructs a unique Nash equilibrium in contests with identical prizes and general nonlinear cost functions. This paper shows that his method, with non-trivial modifications, also applies to contests with asymmetric players and two distinct prizes to show the uniqueness of Nash equilibrium.

Second, this paper provides a closed-form characterization of equilibrium payoffs and strategies in contests with two prizes of arbitrary values. As a result, it unifies the existing equilibrium characterizations with specific prize sequences, and we can illustrate how the unique equilibrium changes from one type to another as the prizes change. In addition, Xiao (2016a) illustrates in an example that a convex prize sequence can lead to an equilibrium in which a player mixes over a non-interval set of performance levels. As a result of the closed-form characterization in this paper, we provide a necessary and sufficient condition for this to happen.

Third, this paper can be used to test conjectures on variants of all-pay auctions and contests as well as on their design questions. If there is significant heterogeneity among either players or prizes (not the limiting cases in which the heterogeneity vanishes), it is typically difficult to characterize equilibria in these games, so it is for related applications. This paper provides a closed-form characterization and computer programs to calculate the strategies and payoffs. Both can be used by researchers to test their conjectures and determine what kind of results to expect. Specifically, we numerically compute the optimal allocation of prizes that maximizes the total expected performance of three asymmetric players. We find that the resulting optimal prize sequence contains either a single prize or two equal prizes. Our findings complement the existing results by examining all the marginal cost profiles in a simplex, including the extreme values of marginal costs that have been previously studied.

Literature There is a large literature on contests and, closely related, auctions. See Konrad (2009) for a comprehensive survey. This paper is closely related to auctions and contests with complete information. As in this paper, Nash equilibria in these setups are usually in mixed strategies. A variety of prize structures are studied. For example, there is a large literature on contests with a single prize (e.g., Baye et al. (1996), Che and Gale (1998)). Identical prizes are considered by Clark and Riis (1998) and Siegel (2009, 2010). Arithmetic prize sequences – with constant first-order differences – are studied by Bulow and Levin (2006) and González-Díaz and Siegel (2013).[1] Xiao (2016a) considers geometric prize sequences, with a constant ratio of two consecutive prizes, and quadratic prize sequences, with constant second-order differences, where both sequences are convex.

The main difference of this paper from the above is that we consider both concave and convex prize sequences. Moreover, we consider how the concavity/convexity affects asymmetric players. Barut and Kovenock (1998) study arbitrary prize sequences in contests among identical players. This paper extends their setup to asymmetric players but restricts it to two distinct prizes. Our findings are different from theirs. We find a unique equilibrium in contrast to their multiple equilibria. In addition, the prize allocation affects the total expected performance in our setup while the total expected performance is independent of prize allocations in their setup.[2] Azmat and Möller (2009) also consider symmetric players in a study of competing contests. Sela (2012) studies sequential all-pay auctions with one object in each stage. Olszewski and Siegel (2016a) study heterogeneous prizes and asymmetric players in large contests where the numbers of prizes

[1] Bulow and Levin (2006) study labor markets in which firms compete for workers. Their model can be transformed into a contest with arithmetic prize sequences.

[2] More precisely, for a fixed budget of prize money, any prize sequence whose lowest prize is zero maximizes the total expected performance.

and players go to infinity. In contrast, this paper considers a similar contest but with a finite number of prizes.

There is a literature on contests with asymmetric information, in contrast to the complete information in this paper. For example, Rosen (1986) studies the role of convex prize sequences in single-elimination tournaments, in which the players' effort is not observable. Moldovanu and Sela (2001) study the optimal allocation of prizes for ex ante symmetric players. All-pay auctions between two ex ante asymmetric players are studied in various setups (e.g., Amann and Leininger (1996), Lizzeri and Persico (2000), Siegel (2014), and Rentschler and Turocy (2016)). However, we cannot study the effects of different prize structures on asymmetric players in those setups because they have either a single prize or symmetric players. Parreiras and Rubinchik (2010, 2015) study all-pay auctions of multiple objects and multiple ex ante asymmetric players. In contrast to this paper, the equilibria in those auctions are in pure strategies.

The remainder of this paper is organized as follows. Section 2 introduces a contest model among three players. Section 3 characterizes the equilibrium payoffs, and Section 4 characterizes the equilibrium strategies. Section 5 studies the optimal prize allocation for asymmetric players. Section 6 generalizes the results to more than three players, and Section 7 concludes.

2 Model

For simpler notation, Sections 2 to 4 focus on a contest with three players $1, 2, 3$. Then, Section 5 extends the results to more players. Each player i has a constant marginal cost of performance $c_i > 0$, and the marginal costs are distinct $0 < c_1 < c_2 < c_3$.[3] Therefore, a performance level $s_i \geq 0$ incurs a cost of $c_i s$ to player i. Player 1 is the strongest because it costs him the least to achieve the same performance. The contest has two distinct prizes $v_1 > v_2 > 0$.[4] Let $\mathbf{c} = (c_1, c_2, c_3)$ be the cost sequence and $\mathbf{v} = (v_1, v_2)$ be the prize sequence. Then, a contest is characterized by $(\mathbf{c}, \mathbf{v})$. The game is of complete information, so $(\mathbf{c}, \mathbf{v})$ is commonly known. Let the first order differences of the prizes be $\Delta_1 = v_1 - v_2$ and $\Delta_2 = v_2 - v_3$, where $v_3 = 0$. Then, the prize sequence is convex if $\Delta_1 > \Delta_2$, linear if $\Delta_1 = \Delta_2$, and concave if $\Delta_1 < \Delta_2$. We use the ratio Δ_1/Δ_2 to measure the convexity of the prize sequence, and we say a sequence is more convex than another if the ratio is larger.

Each player i chooses a performance level $s_i \geq 0$ simultaneously. The player with the highest performance receives the highest prize v_1; the player with the second-highest performance receives the second-highest prize v_2; and the others receive no prize. In the case of a tie, ranks are allocated randomly with equal probabilities to tying players. For example, suppose $s_1 = s_2 > s_3$, then with probability $1/2$, player 1 receives v_1 and player 2 receives v_2; and with probability $1/2$, player 2 receives v_1 and player 1 receives v_2. If $s_1 > s_2 = s_3$, player 2 receives v_2 with probability $1/2$, and player 3 receives v_2 with probability $1/2$. If player i wins prize

[3] If some players have identical marginal costs, there may be multiple Nash equilibria, so our uniqueness result does not apply. See, for instance, Baye et al. (1996).

[4] See Siegel (2010) for the case with $v_1 = v_2$.

v_k with performance s_i, his payoff is $v_k - c_i s_i$; if a player chooses performance $s_i \geq 0$ but wins no prize, his payoff is $-c_i s_i$. All players are risk neutral. We consider only Nash equilibrium throughout the paper.

3 Equilibrium Payoffs

First of all, the contest has no equilibrium in pure strategies, but it has at least one equilibrium in mixed strategies. Siegel (2009) establishes equilibrium existence in contests with identical prizes and his proof is readily extended to our context. As a result, we do not include a proof for equilibrium existence in this paper. Next, we introduce a sequence of definitions, and show in Proposition 1 that the equilibrium payoffs can be constructed using the definitions. After that, Proposition 2 characterizes equilibrium payoffs in closed form, and Corollary 1 discusses comparative statics of the equilibrium payoffs with respect to the prize sequence.

We use a c.d.f. $G_i : [0, +\infty) \to [0, 1]$ to represent player i's mixed strategy. The support of G_i is the smallest closed set to which G_i assigns probability 1. Before the discussion of equilibrium payoffs, we introduce some notation in a two-player contest and a three-player contest.

First, consider a two-player contest in which the top two players 1 and 2 compete for prizes v_1 and v_2. The two-player contest is well-understood.[5] It has a unique equilibrium, and it is in mixed strategies. The equilibrium payoffs are $u_i^2 = v_1 - \Delta_1 c_i/c_2$ for $i = 1, 2$. Throughout the paper, superscripts indicate the number of players in a contest. The equilibrium strategies are $G_1^2(s) = c_2 s/\Delta_1$ and $G_2^2(s) = 1 - c_1/c_2 + c_1 s/\Delta_1$ for $s \in [0, \bar{s}_1^2]$, where $\bar{s}_1^2 = \Delta_1/c_2$.

Second, consider a three-player contest in which player 3 wins in every tie. This contest is the same as the original one described in Section 2 except the tie-breaking rule. More precisely, whenever player 3 has the same performance with another player, player 3 receives a higher prize than the other player.[6] Given linear functions $G_1^2(s) = c_2 s/\Delta_1$ and $G_2^2(s) = 1 - c_1/c_2 + c_1 s/\Delta_1$, define a function $U_3(\cdot|G_1^2, G_2^2) : \mathbb{R} \to \mathbb{R}$ such that

$$U_3(s|G_1^2, G_2^2) \equiv v_1 G_1^2(s) G_2^2(s) + v_2(G_1^2(s)(1 - G_2^2(s)) + (1 - G_1^2(s))G_2^2(s)) - c_3 s$$

which is a quadratic function. The function has an interpretation in the three-player contest in which player 3 wins in every tie. Specifically, if the other players 1 and 2 use strategies G_1^2 and G_2^2, player 3's expected payoffs from choosing s is $U_3(s|G_1^2, G_2^2)$, which is the expected value of his prizes $v_1 G_1^2(s) G_2^2(s) + v_2(G_1^2(s)(1 - G_2^2(s)) + (1 - G_1^2(s))G_2^2(s))$ minus his cost $c_3 s$. Moreover, define

$$\hat{s}_3 \equiv \inf \arg \max_{s \in [0, \bar{s}_1^2]} U_3(s|G_1^2, G_2^2) \tag{1}$$

which is player i's smallest best response against G_1^2 and G_2^2. The expected payoff associated with the best response is

$$\hat{u}_3 \equiv U_3(\hat{s}_3|G_1^2, G_2^2) \tag{2}$$

[5] See, for instance, Che and Gale (2006) and Kaplan and Wettstein (2006).

[6] Without this tie-breaking rule, the interpretation of $U_3(\cdot|G_1^2, G_2^2)$ still applies for $s > 0$ but not for $s = 0$.

and the corresponding expected value of prizes is

$$x_3 \equiv v_1 G_1^2(\hat{s}_3) G_2^2(\hat{s}_3) + v_2(G_1^2(\hat{s}_3)(1 - G_2^2(\hat{s}_3)) + (1 - G_1^2(\hat{s}_3))G_2^2(\hat{s}_3)) = \hat{u}_3 + c_3\hat{s}_3 \tag{3}$$

Now we go back to the two-player contest, in which players 1 and 2 compete for v_1 and v_2, to define x_1 and x_2. Given player 2's equilibrium strategy G_2^2 in this contest, player 1's expected value of prizes from choosing $\hat{s}_3$ defined as in (1) is

$$x_1 \equiv v_1 G_2^2(\hat{s}_3) + v_2(1 - G_2^2(\hat{s}_3)) \tag{4}$$

Similarly, given player 1's strategy G_1^2, player 2's expected value of prizes from choosing $\hat{s}_3$ is

$$x_2 \equiv v_1 G_1^2(\hat{s}_3) + v_2(1 - G_1^2(\hat{s}_3)) \tag{5}$$

With this notation, we can introduce a sequence of definitions that are useful for characterizing equilibrium payoffs.

Definitions

i) Let G_1^2 and G_2^2 be the equilibrium strategies in the two-player contest in which players 1 and 2 compete for prizes v_1 and v_2. Consider the three-player contest in which player 3 wins in every tie. In this contest, define the player 3's *value of winning* as $x_3 = U_3(\hat{s}_3|G_1^2, G_2^2) + c_3\hat{s}_3$, which is his expected value of prizes at his smallest best response $\hat{s}_3$ against other players' strategies G_1^2 and G_2^2.

In the two-player contest with players 1 and 2 and prizes v_1 and v_2, for player $i = 1$ or 2, define his *value of winning* x_i as his expected value of prizes at $\hat{s}_3$ given the other player's equilibrium strategy G_j^2.

ii) The *threshold* T of the contest is the highest performance at which player 3's payoff is zero if his expected value of prizes is x_3, his value of winning.

iii) Player i's *power* w_i is his payoff at the threshold if his expected value of prizes is x_i, his value of winning.

The following example illustrates the above definitions.

Example 1 *Consider a contest of three players with marginal costs $c_1 = 1$, $c_2 = 4$, $c_3 = 7$ and prizes $v_1 = 4$, $v_2 = 3$. Consider the two-player contest in which players 1 and 2 compete for prizes v_1 and v_2. The equilibrium strategies are $G_1^2(s) = 4s$ and $G_2^2(s) = 3/4 + s$. Consider the three-player contest in which player 3 wins in every tie. Suppose players 1 and 2 use strategies G_1^2 and G_2^2, then player 3 has a unique best response $\hat{s}_3 = 0.12$, and the corresponding payoff is $\hat{u}_3 = 2.37$. Players 1 and 2's values of winning are defined in the two-player contest, and they are $x_1 = 3.87$ for player 1 and $x_2 = 3.48$ for player 2 according to (4) and (5). Player 3's value of winning is defined in the three-player contest in which player 3 wins in every tie, and it is $x_3 = \hat{u}_3 + c_3\hat{s}_3 = 3.21$ according to (3). The threshold T satisfies $x_3 - c_3T = 0$, so $T = 0.46$.*

Then, player 1's power is $w_1 = x_1 - c_1T = 3.41$, player 2's power is $w_2 = x_2 - c_2T = 1.64$, and player 3's power is $w_3 = x_3 - c_3T = 0$.

If $v_1 = v_2 = v$, the value of winning in Definition i) is $x_i = v$ for all i, and the definitions of threshold and power are the same as those of Siegel (2010). If $v_1 > v_2$, the definitions are different. In an equilibrium, a player may win v_1, v_2 and 0 with positive probability, so his expected value of prizes may be between v_1 and 0. The "value of winning" in Definition i) takes this into account. As in Example 1, the values of winning are $x_1 = 3.87, x_2 = 3.48$ and $x_3 = 3.21$, all of which are between $v_1 = 4$ and 0. In a continued discussion of Example 1, we show that the value of x_i coincides with player i's expected value of prizes at the threshold T given other players' equilibrium strategies. Using these definitions, the proposition below extends the payoff characterization of Siegel (2009) to contests with heterogeneous prizes.[7]

Proposition 1 *In any equilibrium of the contest, the expected payoff of every player equals the maximum of his power and 0.*

For expositional purposes, all proofs are relegated to the appendix. Proposition 1 transforms the problem of equilibrium payoff characterization – a fixed point problem – into a maximization problem of the quadratic function $U_3(\cdot|G_1^2, G_2^2)$. Therefore, we can use the proposition to find the equilibrium payoffs in closed form. In particular, because $U_3(\cdot|G_1^2, G_2^2)$ is a quadratic function, there are three possible cases for its maximum over the interval $[0, \bar{s}_1^2]$. In Case I, the lower boundary 0 is a maximizer, which may not be the unique maximizer.[8] In Case II, the upper boundary $\bar{s}_1^2$ is the unique maximizer. In Case III, the maximizer is an interior point of $[0, \bar{s}_1^2]$. Case III arises if $U_3'(0|G_1^2, G_2^2) > 0$ and $U_3'(\bar{s}_1^2|G_1^2, G_2^2) < 0$, which are equivalent to

$$c_1 + c_2 < c_3 < 2c_1\Delta_2/\Delta_1 + c_2 - c_1 \tag{6}$$

If (6) does not hold, the maximizers are on the boundaries 0 and $\bar{s}_1^2$. Therefore, Case II happens if $U_3(0|G_1^2, G_2^2) < U_3(\bar{s}_1^2|G_1^2, G_2^2)$, which is equivalent to

$$c_1/(c_3 - c_2) > \Delta_1/\Delta_2 \tag{7}$$

If neither (6) nor (7) holds, Case I arises.

Figure 1 illustrates these conditions for fixed c_1, c_2.[9] In the figure, Case I corresponds to the area with large c_3 and Δ_1/Δ_2. Recall that Δ_1/Δ_2 measures the prize sequence's convexity, so Case I arises with a very weak player 3 and a very convex prize sequence. Case II corresponds to the area with large Δ_1/Δ_2 but small c_3, which means a not too weak player 3 but a very convex prize sequence. Case III corresponds to large c_3 but small Δ_1/Δ_2, which means a very weak

[7] Siegel (2009) also considers very general nonlinear cost functions.

[8] Multiple maximizers arise in two scenarios: First, if $v_1 - 2v_2 > 0$, the objective function $U_3(\cdot|G_1^2, G_2^2)$ is a U-shaped function, whose maximum over $[0, \bar{s}_1^2]$ may be reached at both boundaries of the interval. Second, if $v_1 - 2v_2 = 0$, $U_3(\cdot|G_1^2, G_2^2)$ reduces to a linear function, whose maximum over $[0, \bar{s}_1^2]$ may be reached at every point in the interval.

[9] The curves are plotted for $c_1 = 1, c_2 = 2$.

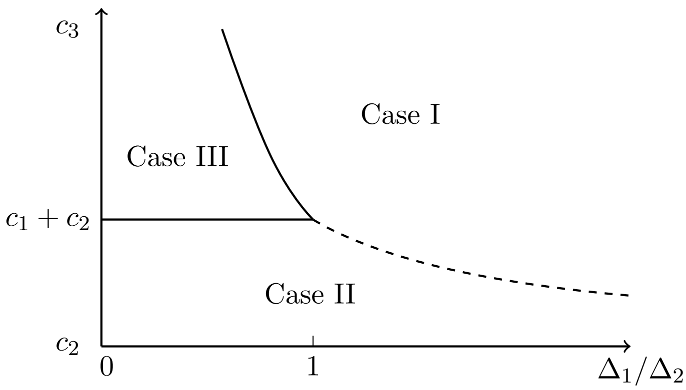

Figure 1: Equilibrium Types

player 3 but not very convex prize sequence. The solid curve corresponds to the upper bound of c_3 in (6), and the dashed curve corresponds to the upper bound of Δ_1/Δ_2 in (7). Moreover, the two curves intersect at $(\Delta_1/\Delta_2, c_3) = (1, c_1 + c_2)$. The proposition below characterizes the equilibrium payoffs for each of the three cases.

Proposition 2
*(**Case I**) If neither (6) nor (7) holds, the equilibrium payoffs are*

$$u_1^* = v_1 - \frac{c_1}{c_2}(v_1 - v_2) - \frac{c_1}{c_3}\left(1 - \frac{c_1}{c_2}\right)v_2,\ u_2^* = \frac{c_3 - c_2 + c_1}{c_3}v_2,\ u_3^* = 0$$

*(**Case II**) If (6) does not hold but (7) does, the equilibrium payoffs are*

$$u_i^* = \left(1 - \frac{c_i}{c_3}\right)v_1 \text{ for } i = 1, 2,\ u_3^* = 0$$

*(**Case III**) If (6) holds, the equilibrium payoffs are*

$$u_1^* = v_1 - \frac{c_1}{c_2}(v_1 - v_2) - \frac{c_1}{c_3}\hat{u}_3,\ u_2^* = v_2 - \frac{c_2}{c_3}\hat{u}_3,\ u_3^* = 0$$

where $\hat{u}_3$ is defined as in (2).

It is worth mentioning that $\hat{u}_3$ is the maximum of a quadratic function that has a closed-form expression in (17). The proposition above implies that the equilibrium payoffs are unique. Recall that we use $(v_1 - v_2)/(v_2 - v_3)$, the ratio of first order differences, to measure the convexity of the prize sequence $\{v_1, v_2, v_3\}$ with $v_3 = 0$. Similarly, we use $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ to measure the convexity of the payoff sequence $\{u_1^*, u_2^*, u_3^*\}$. The result below discusses the comparative statics of the convexity of the payoff sequence with respect to the convexity of the prize sequence.

Corollary 1 *Given $\mathbf{c}$, $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ is nondecreasing in $(v_1 - v_2)/(v_2 - v_3)$. That is, the sequence of equilibrium payoffs is weakly more convex if the prize sequence is more convex.*

4 Equilibrium Strategies

This section characterizes equilibrium strategies in closed form. We first introduce an algorithm to construct a set of strategies. Then, we divide the parameter space of $(\mathbf{c}, \mathbf{v})$ into four subsets. Propositions 3-6 discuss the subsets separately. For each subset, the corresponding proposition characterizes the constructed strategies in closed form and verifies that the strategies are indeed an equilibrium and it is the unique equilibrium. Using the equilibrium payoffs derived in Proposition 2, the algorithm below constructs a strategy profile $\mathbf{G} = (G_1, G_2, G_3)$.

Algorithm:

Step 1. Define $\mathbf{G}$ at the lowest performance $s = 0$: $G_1(0) = G_2(0) = 0, G_3(0) = u_2^*/v_2$.

Step 2. This step examines $\mathbf{G}(0)$ to determine $\mathcal{A}^+(0)$, the set of players whose strategies are increasing at performance $s = 0$. This step contains two parts:

Part One. Define a set of candidates $\mathcal{CP}(s) = \{i \in \{1, 2, 3\}$ such that $U_i(s|G_j, G_k) = u_i^*$ for distinct $i, j, k \in \{1, 2, 3\}\}$.

Part Two. This part refines the candidate set to $\mathcal{A}^+(s)$: Consider an equation system

$$\begin{bmatrix} 0 & K_3(s) & K_2(s) \\ K_3(s) & 0 & K_1(s) \\ K_2(s) & K_1(s) & 0 \end{bmatrix} \begin{bmatrix} g_1(s) \\ g_2(s) \\ g_3(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{8}$$

where $K_i(s) = (\Delta_1 - \Delta_2)G_i(s) + \Delta_2$ for $i = 1, 2, 3$. If the solution $g_3(s) > 0$, let $\mathcal{A}^+(s) = \mathcal{CP}(s)$; otherwise $\mathcal{A}^+(s) = \mathcal{CP}(s)\backslash\{3\}$.

Step 3. Given $\mathbf{G}(0)$ and $\mathcal{A}^+(0)$ defined above, this step extends the definition of $\mathbf{G}$ to performance higher than $s = 0$. Specifically, for $i \in \mathcal{A}^+(s)$ and t slightly higher than s, consider the equation system $U_i(t|G_j, G_k) = u_i^*$ for $j, k \in \mathcal{A}^+(s)\backslash\{i\}$. We can solve $G_i(t) \in [0, 1]$ for $i \in \mathcal{A}^+(s)$ from the system.[10] If $i \notin \mathcal{A}^+(s)$, let $G_i(t) = G_i(s)$. In this way, we can extend $\mathbf{G}$ to performance above s until we reach a switch point s'.

Step 4. This step determines the switch point and extends the definition of $\mathbf{G}$ above the switch point. The switch point s' is the lowest performance above s for which $\mathcal{A}^+(s) \neq \mathcal{A}^+(s')$. This happens for two reasons. The first reason is a player outside of $\mathcal{A}^+(s)$ enters the set at s' according to Step 2. The second reason is a player in $\mathcal{A}^+(s)$ exits the set at s' according to Step 2. Then, given $\mathbf{G}(s')$ and $\mathcal{A}^+(s')$, continue extending $\mathbf{G}$ to higher performance as in Step 3 until another switch point is reached. In this way, we can extend the definition of $\mathbf{G}$ to higher performance until $s = (v_1 - u_1^*)/c_1$ is reached.[11]

The above algorithm is simpler than that of Siegel (2010), which is due to the constant marginal costs. The main difference from his algorithm is Part Two of Step 2. In contrast to Siegel's algorithm, Part Two in our algorithm does not use a fixed point argument. Specifically,

[10] In the proofs of Propositions 3-6, we solve the equation system in four different cases, and show that it has a unique solution.

[11] According to the proofs of Propositions 3-6, there are at most two switch points above 0.

suppose $\mathcal{CP}(s) = \{1,2,3\}$ from Part One. Then, the definition of $\mathcal{CP}(s)$ implies $U_i(s|G_j, G_k) = u_i^*$ for $j,k \in \{1,2,3\}\backslash\{i\}$. If all candidates' strategies in an equilibrium are indeed increasing at s, the above equation remains true for performance levels slightly above s. Differentiating both sides of the equation with respect to s, we obtain $K_j(s)g_k(s) + K_k(s)g_j(s) = c_i$, where $g_i(s)$ is the derivative of G_i.[12] The matrix form of these equations is (8) in Part Two. Therefore, if all candidates' strategies are indeed increasing at s in an equilibrium, (8) must have positive solutions $g_i(s) > 0$ for $i = 1,2,3$. If $g_3(s) \leq 0$, then at least one candidate's strategy is not increasing at s. It turns out this candidate must be player 3, who has the highest marginal cost. So far we explain Part Two in the case of $\mathcal{CP}(s) = \{1,2,3\}$ and $g_3(s) \leq 0$, and this case turns out to be the only case in which $\mathcal{CP}(s)$ is different from $\mathcal{A}^+(s)$.[13]

We verify below that the mixed strategies constructed by the algorithm determine the unique equilibrium. Based on the supports of the mixed strategies, we categorize the strategies into four types: Type I, Type II, Type II$'$, and Type III. The equilibrium of Type I has payoffs as in Case I, the equilibrium of Types II or II$'$ has payoffs as in Case II, and the equilibrium of Type III has payoffs as in Case III. We discuss the four types in four propositions.

Proposition 3 (Type I) *If $\Delta_1/\Delta_2 \geq 1$ and $\Delta_1/\Delta_2 \geq c_1/(c_3 - c_2)$ or if $2c_1/(c_1 + c_3 - c_2) \leq \Delta_1/\Delta_2 \leq 1$, the equilibrium payoffs are as in Case I, and the strategies in the unique equilibrium are*

$$
\begin{aligned}
G_1^*(s) &= s\frac{c_2}{\Delta_1} - \frac{c_2 - c_1}{c_3}\frac{\Delta_2}{\Delta_1} \quad \text{for } s \in \left[\frac{\Delta_2(c_2 - c_1)}{c_2c_3}, \frac{\Delta_2(c_2 - c_1)}{c_2c_3} + \frac{\Delta_1}{c_2}\right] \\
G_2^*(s) &= \begin{cases} s\dfrac{c_3}{\Delta_2} & \text{for } s \in \left[0, \dfrac{\Delta_2(c_2 - c_1)}{c_2c_3}\right] \\ s\dfrac{c_1}{\Delta_1} + \left(1 - \dfrac{c_1}{c_2}\right)\left(1 - \dfrac{c_1}{c_3}\dfrac{\Delta_2}{\Delta_1}\right) & \text{for } s \in \left[\dfrac{\Delta_2(c_2 - c_1)}{c_2c_3}, \dfrac{\Delta_2(c_2 - c_1)}{c_2c_3} + \dfrac{\Delta_1}{c_2}\right] \end{cases} \\
G_3^*(s) &= s\frac{c_2}{\Delta_2} + \frac{c_3 - c_2 + c_1}{c_3} \quad \text{for } s \in \left[0, \frac{\Delta_2(c_2 - c_1)}{c_2c_3}\right]
\end{aligned}
$$

The proofs of Propositions 3-6 are in the appendix. In each proof, we first show that the algorithm constructs a unique set of strategies and derive their close-form characterization. In particular, we explain how to solve the equation system in Step 3 and why the solution is unique. Then, we verify that the constructed strategies are indeed an equilibrium. After that, we show that every equilibrium must be one of the outcomes of the algorithm, which, combined with the unique outcome of the algorithm, implies equilibrium uniqueness.

If the unique equilibrium is of Type I, there are two intervals $[0, \Delta_2(c_2 - c_1)/(c_2c_3)]$ and $[\Delta_2(c_2 - c_1)/(c_2c_3), \Delta_2(c_2 - c_1)/(c_2c_3) + \Delta_1/c_2]$. Player 1 mixes over the higher interval, player 3 mixes over the lower one, and player 2 mixes over both. For example, if $v_1 = 3, v_2 = 1$ and $c_1 = 1, c_2 = 4, c_3 = 7$, the contest has a Type I equilibrium. Figure 2 illustrates the equilibrium strategies.

[12] The derivatives exist because of the implicit function theorem.

[13] This is because of the "Nested Gaps" property in Lemma 4.

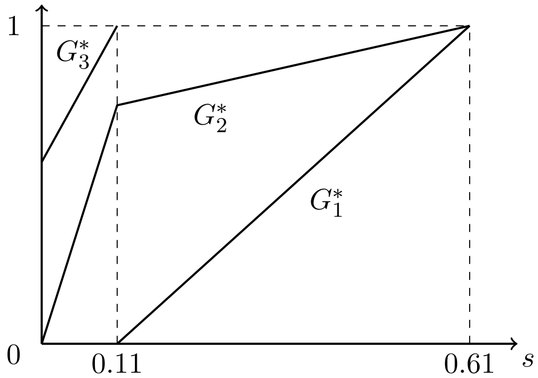

Figure 2: Type I Equilibrium

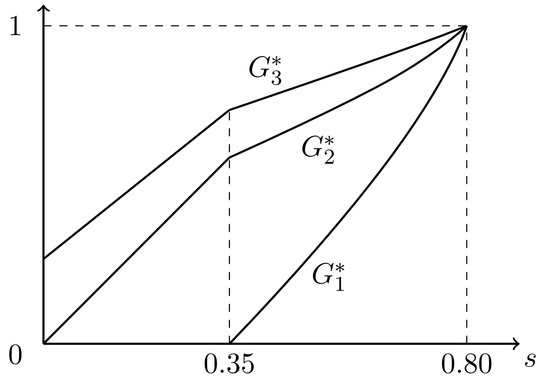

Figure 3: Type II Equilibrium

For the Type II equilibrium, define $A_i = u_i^* + (\Delta_2)^2/(\Delta_1 - \Delta_2)$ for $i = 1, 2, 3$, and $\underline{s}_1^*$ as the smallest $s \geq 0$ such that

$$(v_1 - 2v_2)\frac{c_3 s}{v_2}\frac{u_2^* + c_2 s}{v_2} + v_2\left(\frac{c_3 s}{v_2} + \frac{u_2^* + c_2 s}{v_2}\right) - c_3 s = u_1^*$$

In addition, if $\Delta_1 \neq \Delta_2$, for $i = 1, 2, 3$ define

$$F_i(s) = \frac{1}{c_i s + A_i}\sqrt{\frac{\times_{j=1}^{3}(c_j s + A_j)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$$

which is repeatedly used in the characterization of the Type II, II' and III equilibria.[14]

Proposition 4 (Type II) *If $\Delta_1/\Delta_2 > 1$, $\Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and*

$$\frac{\partial}{\partial s}\left(\frac{(A_1 + c_1 s)(A_2 + c_2 s)}{A_3 + c_3 s}\right)\Bigg|_{\underline{s}_1^*} \geq 0 \tag{9}$$

or if $\Delta_1/\Delta_2 < 1$ and $c_3 \leq c_1 + c_2$, the equilibrium payoffs are as in Case II and the strategies in the unique equilibrium are

$$\begin{aligned}
G_1^*(s) &= F_1(s) \quad \text{for } s \in [\underline{s}_1^*, v_1/c_3] \\
G_2^*(s) &= \begin{cases} sc_3/\Delta_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_2(s) & \text{for } s \in [\underline{s}_1^*, v_1/c_3] \end{cases} \\
G_3^*(s) &= \begin{cases} sc_2/v_2 + (1 - c_2/c_3)\, v_1/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_3(s) & \text{for } s \in [\underline{s}_1^*, v_1/c_3] \end{cases}
\end{aligned}$$

If $\Delta_1/\Delta_2 = 1$ and $c_3 < c_1 + c_2$, the strategies are the same except that $G_i^*(s) = [\sum_{j=1}^{3}(u_j^* + c_j s) - 2(u_i^* + c_i s)]/(2v_2)$ for $i = 1, 2, 3$ and for $s \in [\underline{s}_1^*, v_1/c_3]$.

In an equilibrium of Type II, all three strategies have interval supports, and their supports share the same upper boundary. For example, if $v_1 = 4, v_2 = 3$ and $c_1 = 2, c_2 = 4, c_3 = 5$, the contest has a Type II equilibrium. Figure 3 illustrates the equilibrium strategies.

[14] Under the conditions of Propositions 4-6, $F_i(s)$ is a real number. See, for example, the discussion below (23).

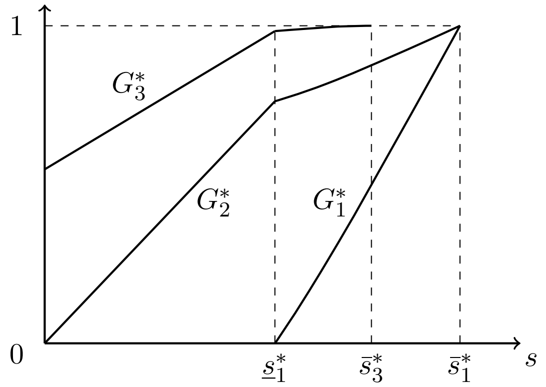

Figure 4: Type III Equilibrium

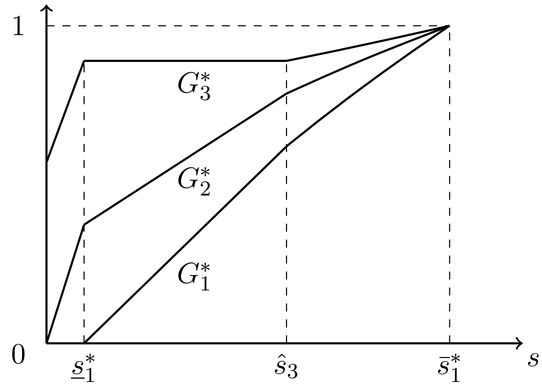

Figure 5: Type II′ Equilibrium

Proposition 5 (Type III) *If $c_1 + c_2 < c_3 < (c_2 - c_1)(\Delta_1 - \Delta_2)/\Delta_1 + (c_1 + c_2)\Delta_2/\Delta_1$, the equilibrium payoffs are as in Case III, and the strategies in the unique equilibrium are*

$$G_1^*(s) = \begin{cases} F_1(s) & \text{for } s \in [\underline{s}_1^*, \bar{s}_3^*] \\ (c_2 s + u_2^* - v_2)/\Delta_1 & \text{for } s \in [\bar{s}_3^*, \bar{s}_1^*] \end{cases}$$

$$G_2^*(s) = \begin{cases} sc_3/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_2(s) & \text{for } s \in [\underline{s}_1^*, \bar{s}_3^*] \\ (sc_1 + u_1^* - v_2)/\Delta_1 & \text{for } s \in [\bar{s}_3^*, \bar{s}_1^*] \end{cases}$$

$$G_3^*(s) = \begin{cases} (sc_2 + u_2^*)/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_3(s) & \text{for } s \in [\underline{s}_1^*, \bar{s}_3^*] \end{cases}$$

where $\underline{s}_1^*$ is defined before Proposition 4, $\bar{s}_1^* = (v_1 - u_1^*)/c_1$, and $\bar{s}_3^*$ is the smallest $s \geq \underline{s}_1^*$ such that $F_3(s) = 1$.

The following example illustrates an equilibrium of Type III.

Example 1 (continued) *Consider the contest in Example 1. The equilibrium payoffs are $u_1^* = 3.41$ for player 1, $u_2^* = 1.64$ for player 2, and $u_3^* = 0$ for player 3. All players' mixed strategies have interval supports. The supports are $[0, 0.46]$ for player 3, $[0, 0.59]$ for player 2, and $[0.32, 0.59]$ for player 1. Figure 4 illustrates the equilibrium strategies. Two properties are worth mentioning: First, recall that the threshold of this contest is $T = 0.46$, which is exactly the highest performance in the support of 3's equilibrium strategy. Second, we can verify that player i's "value of winning", x_i, equals the expected value of his prize at the threshold given others' equilibrium strategies. Hence, player i's equilibrium payoff is $u_i^* = x_i - c_i T$.*[15]

Proposition 6 (Type II′) *If we have $\Delta_1/\Delta_2 > 1$, $\Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ but not (9), the*

[15]Both properties are true in general. We can verify the properties using the closed-form characterization of equilibrium strategies.

equilibrium payoffs are as in Case II, and the strategies in the unique equilibrium are

$$
\begin{aligned}
G_1^*(s) &= \begin{cases} \dfrac{c_2 s + u_2^* - v_2 G_3^*(\underline{s}_1^*)}{(\Delta_1 - \Delta_2) G_3^*(\underline{s}_1^*) + v_2} & \textit{for } s \in [\underline{s}_1^*, \hat{s}_3] \\ F_1(s) & \textit{for } s \in [\hat{s}_3, \bar{s}_1^*] \end{cases} \\
G_2^*(s) &= \begin{cases} s c_3 / \Delta_2 & \textit{for } s \in [0, \underline{s}_1^*] \\ \dfrac{c_1 s + u_1^* - v_2 G_3^*(\underline{s}_1^*)}{(\Delta_1 - \Delta_2) G_3^*(\underline{s}_1^*) + v_2} & \textit{for } s \in [\underline{s}_1^*, \hat{s}_3] \\ F_2(s) & \textit{for } s \in [\hat{s}_3, \bar{s}_1^*] \end{cases} \\
G_3^*(s) &= \begin{cases} s c_2 / v_2 + (1 - c_2/c_3)\, v_1/v_2 & \textit{for } s \in [0, \underline{s}_1^*] \\ \underline{s}_1^* c_2 / v_2 + (1 - c_2/c_3)\, v_1/v_2 & \textit{for } s \in [\underline{s}_1^*, \hat{s}_3] \\ F_3(s) & \textit{for } s \in [\hat{s}_3, \bar{s}_1^*] \end{cases}
\end{aligned}
$$

where $\hat{s}_3$ is the smallest $s \geq \underline{s}_1^*$ such that $U_3(s|F_1, F_2) = u_3^*$.

Proposition 6 provides a necessary and sufficient condition on the convexity of the prize sequence for equilibrium strategies with non-interval supports. This complements Siegel's (2010) result that, in contests with identical prizes, linear cost functions result in equilibrium strategies with interval supports, but nonlinear cost functions may not. The following example illustrates an equilibrium of Type II′.

Example 2 *Consider a contest of three players with marginal costs $c_1 = 4$, $c_2 = 6$, $c_3 = 7$ and prizes $v_1 = 4$, $v_2 = 1$. The equilibrium payoffs are $u_1^* = 1.71$ for player 1, $u_2^* = 0.57$ for player 2, and zero for player 3. Player 1 mixes over interval $[0.05, 0.57]$, player 2 mixes over $[0, 0.57]$, and player 3 mixes over $[0, 0.05] \cup [0.34, 0.57]$. The equilibrium strategies are*[16]

$$
\begin{aligned}
G_1^*(s) &= \begin{cases} 2.16s - 0.11 & \textit{for } s \in [0.05, 0.34] \\ \sqrt{0.5(84s + 15)(14s + 1)/(56s + 31)} - 0.5 & \textit{for } s \in [0.34, 0.57] \end{cases} \\
G_2^*(s) &= \begin{cases} 7s & \textit{for } s \in [0, 0.05] \\ 1.44s + 0.30 & \textit{for } s \in [0.05, 0.34] \\ \sqrt{0.5(56s + 31)(14s + 1)/(84s + 15)} - 0.5 & \textit{for } s \in [0.34, 0.57] \end{cases} \\
G_3^*(s) &= \begin{cases} 6s + 0.57 & \textit{for } s \in [0, 0.05] \\ 0.89 & \textit{for } s \in [0.05, 0.34] \\ \sqrt{0.5(84s + 15)(56s + 31)/(14s + 1)} - 0.5 & \textit{for } s \in [0.34, 0.57] \end{cases}
\end{aligned}
$$

Given G_1^* and G_2^*, player 3's payoff from choosing $s \in (0.05, 0.34)$ is a U-shaped quadratic curve passing 0 at the boundaries of the interval, so the payoff is lower than 0. Figure 5 illustrates the equilibrium strategies.

Combining Propositions 3-6, we have the following result.

[16] The slopes and intercepts of the linear parts of the strategies are rounded to two decimal places.

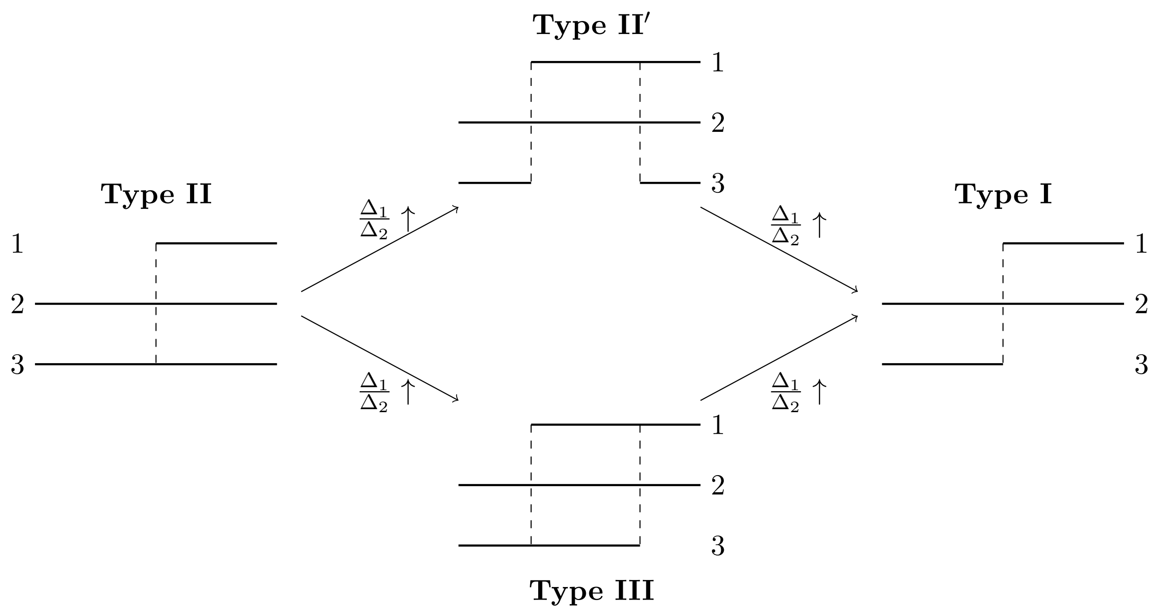

Figure 6: Supports of Mixed Strategies

Corollary 2 *The contest has a unique Nash equilibrium, and it is constructed by the algorithm.*

According to Propositions 3-6, the unique equilibrium can be one of four types. These propositions unify the existing equilibrium characterizations for different specific prize sequences. For example, Bulow and Levin (2006) illustrate the equilibrium of Type I or II if the prize sequence is arithmetic. Siegel (2010) shows the equilibrium of Type II if the prizes are identical. Xiao (2016a) shows the equilibrium of Type II′ if the prize sequence is convex. To our knowledge, the Type III equilibrium has not been discussed in the literature. By adding Type III, this paper provides a complete list of equilibrium types.

In addition, we can illustrate how the different equilibrium types relate to each other. More precisely, we consider how the unique equilibrium changes from one type to another as the prize sequence becomes more convex.

Proposition 7 *If $c_3 > c_1 + c_2$, the equilibrium is of Type II for $\Delta_1/\Delta_2 = 0$; Type III for $\Delta_1/\Delta_2 \in (0, 2c_1/(c_1 + c_3 - c_2))$; and Type I for $\Delta_1/\Delta_2 > 2c_1/(c_1 + c_3 - c_2)$.*

If $c_3 \leq c_1 + c_2$, there exists $\lambda \in (1, c_1/(c_3 - c_2))$ such that the equilibrium is of Type II for $\Delta_1/\Delta_2 < \lambda$, Type II′ for $\Delta_1/\Delta_2 \in (\lambda, c_1/(c_3 - c_2))$; and Type I for $\Delta_1/\Delta_2 \geq c_1/(c_3 - c_2)$.

The transition is demonstrated in Figure 6. In the figure, we use the supports of the equilibrium strategies to demonstrate different types. If Δ_1/Δ_2 is small, the equilibrium is of Type II; if Δ_1/Δ_2 is large enough, the equilibrium is of Type I. How does the equilibrium transform from Type II to Type I as Δ_1/Δ_2 increases? If $c_3 > c_1 + c_2$, the equilibrium changes from Type II to Type II′ then to Type I, which is illustrated at the upper half of the figure. If $c_3 \leq c_1 + c_2$, the equilibrium changes from Type II to Type III then to Type I, which is illustrated at the lower half of the figure.

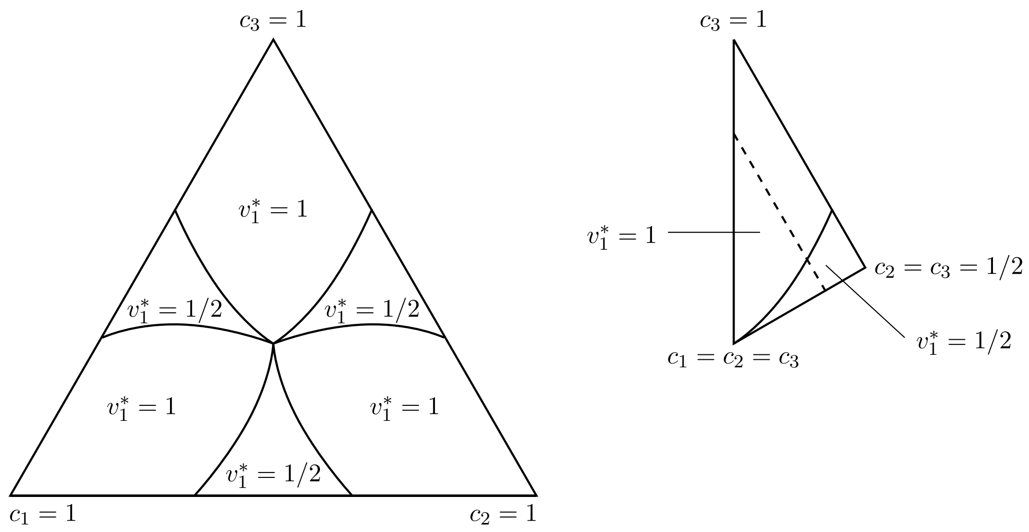

Figure 7: Performance Maximizing Prizes

We show above the unique equilibrium in contests with linear cost functions and two distinct prizes. We describe below two potential extensions and what we know about them. First, the method in this paper can be extended to show equilibrium uniqueness if the cost functions are nonlinear and have ordered marginal cost functions. However, the equilibrium generally has no closed-form characterization. Another extension is to consider more than two positive prizes. However, equilibrium strategies generally do not have a closed-form characterization either.[17]

5 Performance Maximizing Prizes

Consider a contest organizer with a fixed budget whose value is normalized to 1. She is risk neutral and wants to maximize the total expected performance by splitting the budget into v_1 and v_2 such that $v_1, v_2 \geq 0$ and $v_1 + v_2 = 1$. Because there are three players, it is not optimal to have three or more prizes. Because of the closed-form characterization of the equilibrium strategies, we can numerically calculate the expected performance given each prize allocation. Therefore, we can find the optimal prize allocation (v_1^*, v_2^*) given any cost profile $\mathbf{c} = (c_1, c_2, c_3)$.

In Figure 7, the triangle on the right illustrates the optimal prize allocation for $\mathbf{c} = (c_1, c_2, c_3)$ satisfies $c_1 + c_2 + c_3 = 1$ and $0 \leq c_1 \leq c_2 \leq c_3$. For a given c_1, the dashed line represents $\mathbf{c}$ such that $c_2 + c_3 = 1 - c_1$. Moreover, moving up along the dashed line leads to higher values of $(c_3 - c_2)/(c_2 - c_1)$. Therefore, we have two key observations from the figure:

- *The optimal allocation* (v_1^*, v_2^*) *is either* $(1/2, 1/2)$ *or* $(1, 0)$.
- *For a given* c_1*, there exists* $\phi \in (0, \infty)$ *such that the optimal allocation is* $(v_1^*, v_2^*) = (1/2, 1/2)$ *if* $(c_3 - c_2)/(c_2 - c_1) < \phi$*, and* $(v_1^*, v_2^*) = (1, 0)$ *if* $(c_3 - c_2)/(c_2 - c_1) > \phi$.

[17] See Xiao (2016a) for examples of such equilibria.

This means that a single prize – the most convex prize sequence – maximizes the total expected performance in a three-player contest if the top two players are similar, and two equal prizes – the most concave prize sequence – maximize the total expected performance if the bottom two players are similar. Moreover, according to Corollary 1, if the top two players are similar, the performance maximizing prizes result in the most convex equilibrium payoff sequence; if the bottom two players are similar, the performance maximizing prizes result in the least convex equilibrium payoff sequence. By renaming the players, the above results extend to the entire simplex in which $\mathbf{c} = (c_1, c_2, c_3)$ satisfies $c_1 + c_2 + c_3 = 1$ and $c_i \geq 0$ for $i = 1, 2, 3$. In Figure 7 the triangle on the left illustrates the optimal prize allocation over the entire simplex.[18]

There is a big literature on contests with multiple prizes (see Sisak (2009) for a survey), and multiple prizes are studied in various scenarios, e.g. contests with participation constraints (Megidish and Sela (2013)). Our analysis above focuses on how the asymmetry of players' costs affects the optimal allocation of prizes. In a different setup with ex ante symmetric players, Moldovanu and Sela (2001) study how the convexity of the players' cost function affects the optimal prize allocation. They show that a single prize is optimal if the players have concave or linear cost functions, and multiple prizes are optimal if they have concave cost functions. In contrast, our results suggest that in the case of complete information, multiple prizes can be optimal even if the cost functions are linear.

The optimality of multiple prizes is also demonstrated in various limiting cases. For example, Szymanski and Valletti (2005) and Xiao (2016a) consider contests in which the strongest player's marginal cost converges to zero. Cohen and Sela (2008) study an all-pay auction in which one player values the second prize slightly higher than the other players. Our findings complement those results by examining all the cost profiles in the simplex, including the extreme values of marginal costs.

6 More Than Three Players

Consider the same contest described in Section 2 except that there are more than three players: $1, 2, ..., n$, where $n \geq 3$.[19] They have constant marginal costs of performance, which satisfy $0 < c_1^n < c_2^n < ... < c_n^n$. The following result extends the closed form characterization of equilibrium payoffs and strategies to the n-player contest.

Proposition 8 *The n-player contest has a unique equilibrium. The players $1, 2, 3$'s payoffs are the same as their payoffs in the three-player contest, the other players' payoffs are zero. Moreover, the players $1, 2, 3$'s equilibrium strategies are the same as their equilibrium strategies in the three-player contest, the other players choose $s = 0$ with probability 1.*

Next, we consider the performance maximizing prizes for $n > 3$ players. Olszewski and Siegel (2016b) examine performance maximizing prizes in large contests in which the number of players

[18] In a zero-measure subset of the simplex, $\mathbf{c}$ contains identical marginal costs. For those values of $\mathbf{c}$, our method still constructs an equilibrium, which we use in the simulation. However, there may be other equilibria.

[19] The two-player contest is well understood in the literature. See Footnote 8.

goes to infinity. They find that one prize is optimal if the limiting distribution of the players' constant marginal costs has a continuous and strictly positive density.[20] Our finding is different from theirs. Specifically, for $n > 6$, consider an n-player contest with $c_1^n = 1/n, c_2^n = 1/2 - 2/n$ and $c_i^n = 1/2 + (i-2)/n$ for $i \geq 3$. Note that $0 < c_1^n < c_2^n < ... < c_n^n$ and $c_1^n + c_2^n + c_3^n = 1$. Given one or two prizes, players $4, ..., n$ choose zero performance with probability 1, so the total expected performance in this contest is the same as the three-player contest among players 1, 2 and 3. Notice that $\lim_{n \to +\infty} c_3^n - c_2^n = 0$, so our observations above suggest that, for a large enough n, two equal prizes result in higher expected performance than one prize in the n-player contests. The main reason for the difference is that, if $n \to +\infty$, the limiting distribution of the marginal costs is $F(c) = 0$ for $c \in [0, 1/2]$ and $F(c) = c - 1/2$ for $c \in [1/2, 3/2]$. Its density function over $(0, 1/2)$ is 0, which is excluded by Olszewski and Siegel (2016b).

7 Conclusion

This paper studies a contest in which three or more players with asymmetric costs compete for two nonidentical prizes. The prize sequence can be either concave or convex. We show that the equilibrium in this contest is unique, and provide a closed-form characterization of equilibrium payoffs and strategies. In addition, we show how the convexity of the prize sequence affects the equilibrium strategies and payoffs.

A computer program is also provided to calculate the equilibrium strategies and payoffs. The closed-form characterization and the computer program are useful to study contest design questions. As an example, we compute the optimal allocation of prizes that maximizes the total expected performance in a contest of asymmetric players, and illustrate the convexity of the resulting optimal prize sequence.

References

Amann, E. and Leininger, W. (1996), "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case", *Games and Economic Behavior*, 14, pp. 118. [4]

Azmat, G. and Möller, M. (2009), "Competition among Contests", *RAND Journal of Economics*, 40, pp. 743-768. [3]

Barut, Y. and Kovenock, D. (1998), "The Symmetric Multiple Prize All-Pay Auction with Complete Information", *European Journal of Political Economy*, 14, pp. 627-644. [3]

Baye, M. R.; Kovenock, D. and de Vries, C. G. (1996), "The All-Pay Auction with Complete Information", *Economic Theory*, 8, pp. 291-305. [2, 3, and 4]

Bulow, J. and Levin, J. (2006), "Matching and Price Competition", *American Economic Review*, 96, pp. 652-668. [3, 14, and 19]

[20]See their Proposition 7.

Clark, D. and Riis, C. (1998), “Competition over More than One Prizes”, *American Economic Review*, 88. pp. 276-289. [3]

Che, Y. and Gale, I. (1998), “Caps on Political Lobbying”, *American Economic Review*, 88, pp. 643-651. [3]

Che, Y. and Gale, I. (2006), “Caps on Political Lobbying: Reply”, *American Economic Review*, 96, pp. 1355-1360. [5]

Cohen, C. and Sela, A. (2008), “Allocation of Prizes in Asymmetric All-pay Auctions”, *European Journal of Political Economy*, 24, pp. 123-132. [16]

González-Díaz, J. and Siegel, R. (2013), “Matching and Price Competition: Beyond Symmetric Linear Costs”, *International Journal of Game Theory*, 42, pp. 835-844. [3]

Kaplan, T. R. and Wettsstein, D. (2006), “Caps on Political Lobbying: Comment”, *American Economic Review*, 96, pp. 1351-1354. [5]

Konrad, K. A. (2009), *Strategy and Dynamics in Contests*, Oxford University Press. [3]

Lizzeri, A. and Persico, N. (2000), “Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price”, *Games and Economic Behavior*, 30, pp. 83-114. [4]

Megidish, R. and Sela, A. (2013), “Allocation of Prizes in Contests with Participation Constraints”, *Journal of Economics and Management Strategy*, 22, pp. 713-727. [16]

Moldovanu, B. and Sela, A. (2001), “The Optimal Allocation of Prizes in Contests,” *American Economic Review*, 91, pp. 542-558. [4 and 16]

Olszewski, W. and Siegel, R. (2016a), “Large Contests”, *Econometrica*, 84, pp. 835-854. [3]

Olszewski, W. and Siegel, R. (2016b), “Effort-Maximizing Contests”, Mimeo, Northwestern University. [16 and 17]

Parreiras, S. O. and Rubinchik, A. (2010), “Contests with Three or More Heterogeneous Agents,” *Games and Economic Behavior*, 68, pp. 703–715. [4]

Parreiras, S. O. and Rubinchik, A. (2015), “Group Composition in Contest,” Mimeo. [4]

Rentschler, L. and Turocy, T. L. (2016), “Two-Bidder All-Pay Auctions with Interdependent Valuations, Including the Highly Competitive Case”, *Journal of Economic Theory,* 163, pp. 435-466. [4]

Rosen, S. (1986), “Prizes and Incentives in Elimination Tournaments”, *American Economic Review,* 76, pp. 701-715. [4]

Sela, A. (2012), “Sequential Two-Prize Contests”, *Economic Theory,* 51, pp. 383-395. [3]

Siegel, R. (2009), "All-Pay Contests", *Econometrica,* 77, pp. 71-92. [3, 5, and 7]

Siegel, R. (2010), "Asymmetric Contests with Conditional Investments", *American Economic Review*, 100, pp. 2230-2260. [2, 3, 4, 7, 9, 13, and 14]

Siegel, R. (2014), "Asymmetric All-Pay Auctions with Interdependent Valuations", *Journal of Economic Theory,* 153, pp. 684-702. [4]

Sisak, D. (2009), "Multiple-Prize Contests - the Optimal Allocation of Prizes", *Journal of Economic Surveys,* 23, pp. 82-114. [16]

Szymanski, S. and Valletti, T. (2005), "Incentive Effects of Second Prizes", *European Journal of Political Economy*, 21, pp. 467-481. [16]

Xiao, J. (2016a), "Asymmetric All-Pay Contests with Heterogeneous Prizes", *Journal of Economic Theory*, 163, pp. 178-221. [3, 14, 15, 16, and 19]

Xiao, J. (2016b), "Ability Grouping in Contests", Mimeo, University of Melbourne. [19]

Appendix

We first introduce several known equilibrium properties in Lemmas 1 to 5, then some additional properties in Lemmas 6 and 7. See Step 2 in Appendix A of Bulow and Levin (2006) for Lemma 1, Xiao (2016b) for Lemma 2, and Xiao (2016a) for Lemmas 3-5. These papers consider different prize sequences, but their proofs apply here. After that, we use the lemmas to prove Proposition 1. For an equilibrium with strategies $G_1^*, ..., G_n^*$ in the n-player contest, let $\underline{s}_i^*$ and $\bar{s}_i^*$ be the minimal and maximum performance in the support of G_i^*.

Lemma 1 ***(No Aggregate Gaps)*** *In every equilibrium, if* $s \in [0, \max(\bar{s}_1^*, ..., \bar{s}_n^*)]$*, there are at least two players whose equilibrium strategies' supports contain* s*.*

This means the distribution of all players' performance must have an interval support.

Lemma 2 *Suppose a player has an atom at performance level* s *in an equilibrium, that is, he chooses* s *with positive probability. Then, he receives no prize by choosing* s*.*

According to the lemma, a player never has an atom at $s > 0$ in an equilibrium, otherwise he receives a negative expected payoff. Therefore, the possible atoms in an equilibrium must be at $s = 0$.

Lemma 3 ***(Participation)*** *In every equilibrium, player* $i > 3$ *assigns probability 1 to* $s = 0$*.*

The above lemma implies that the top three players, whose costs are the lowest, choose positive performance to compete for the two prizes. The other players' costs are too high and they give up by choosing $s = 0$.

Lemma 4 ***(Nested Gaps)*** *In every equilibrium, if $s \in (\underline{s}_i^*, \bar{s}_i^*)$ is not in the support of G_i^*, then s is not in the support of G_j^* for any $j > i$.*

If $s \in (\underline{s}_i^*, \bar{s}_i^*)$ is not in the support of G_i^*, it means that, due to the competition from other players, s is not a best response for player i. Then, Lemma 4 implies that for the players weaker than i, the performance s is not a best response either.

Lemma 5 ***(Stochastic Dominance)*** *In every equilibrium, if $i < j$, then $G_i^*(s) \leq G_j^*(s)$ for $s \geq 0$.*

If $i < j$, player i is stronger than j because i has a lower marginal cost. Then, the lemma means the equilibrium performance levels of a stronger player are higher, in terms of the first order stochastic dominance, than those of a weaker player.

Lemma 6 *In every equilibrium, players 1 and 2's strategies G_1^* and G_2^* have no atoms; player 3's strategy G_3^* has an atom at $s = 0$; and player $i \geq 4$ assigns probability 1 to $s = 0$.*

Proof. We prove this in three steps. First, Lemma 3 implies that player $i \geq 4$ assigns probability 1 to $s = 0$. As a result, given others' equilibrium strategies, a player's payoff at a positive performance level does not depend on the strategies of $i \geq 4$.

Second, G_1^* and G_2^* have no atoms. Suppose otherwise that in an equilibrium, G_1^* has an atom. Then, Lemma 2 implies that player 1's expected equilibrium payoff $u_1^* = 0$. Recall that $\bar{s}_3^*$ is the highest performance in the support of G_3^*. If $\bar{s}_3^* = 0$, player 1 could receive a positive expected payoff by deviating to $s = 0$. Therefore, $\bar{s}_3^* > 0$. Due to Lemma 2, $\bar{s}_3^* > 0$ cannot be an atom. Hence, given others' equilibrium strategies, player 1's expected payoff at $\bar{s}_3^*$ is $u_1^* = U_1(\bar{s}_3^* | G_2^*, G_3^*)$, where

$$U_i(s|G_j, G_k) = v_1 G_j(s) G_k(s) + v_2(G_j(s)(1 - G_k(s)) + (1 - G_j(s))G_k(s)) - c_i s$$

Notice that player 1's payoff at $\bar{s}_3^*$ is independent of G_i^* for $i \geq 4$, which is due to the first step. Similarly, $u_3^* = U_3(\bar{s}_3^* | G_2^*, G_1^*)$. In addition, because $G_3^*(\bar{s}_3^*) = 1 \geq G_1^*(\bar{s}_3^*)$ and $c_3 \bar{s}_3^* > c_1 \bar{s}_3^*$, we have $U_1(\bar{s}_3^* | G_2^*, G_3^*) > U_3(\bar{s}_3^* | G_2^*, G_1^*)$. As a result, $u_1^* > u_3^* \geq 0$, which contradicts $u_1^* = 0$.

Third, G_3^* has an atom at $s = 0$. Suppose otherwise that $G_3^*(0) = 0$. Recall that $\underline{s}_i^*$ is the smallest performance in the support of G_i^*. Then, if $\min(\underline{s}_1^*, \underline{s}_2^*) > 0$, the interval $(0, \min(\underline{s}_1^*, \underline{s}_2^*))$ receives zero probability from G_i^* for $i \neq 3$. This contradicts Lemma 1, the property of "No Aggregate Gaps". Therefore, $\min(\underline{s}_1^*, \underline{s}_2^*) = 0$. Without loss of generality, assume $\underline{s}_2^* = 0$. Recall that $G_1^*(0) = 0$ according to the second step. Then, the assumption $G_3^*(0) = 0$ implies that, given others' equilibrium strategies, player 2's expected payoff at $\underline{s}_2^* = 0$ is zero. Therefore, $u_2^* = 0$, which leads to a contradiction by the same argument in the second step. Hence, G_3^* has an atom at $s = 0$. ■

Lemma 7 *In every equilibrium, for any* $s \in (\underline{s}_1^*, \bar{s}_3^*)$, $\bar{G}_1(s) < G_1^*(s)$ *and* $\bar{G}_2(s) < G_2^*(s)$, *where* $\bar{G}_1(s), \bar{G}_2(s)$ *solve*

$$\begin{aligned} v_1\bar{G}_2(s) + v_2(1 - \bar{G}_2(s)) - c_1 s &= u_1^* && (10) \\ v_1\bar{G}_1(s) + v_2(1 - \bar{G}_1(s)) - c_2 s &= u_2^* && (11) \end{aligned}$$

Proof. According to Lemma 6, any player $i \geq 4$ chooses zero with probability 1. As a result, Lemma 1, the property of "No Aggregate Gaps", implies that at least two of players 1, 2, 3's strategies' supports contain $s \in (\underline{s}_1^*, \bar{s}_3^*)$, and Lemma 4, the property of "Nested Gaps", implies that 1 and 2 must be among these players. Therefore,

$$\begin{aligned} U_1(s|G_2^*, G_3^*) &= u_1^* && (12) \\ U_2(s|G_1^*, G_3^*) &= u_2^* && (13) \end{aligned}$$

By the definition of $\bar{s}_3^*$, we have $G_3^*(s) < 1$ for $s < \bar{s}_3^*$. Notice that (12) and (13) implicitly define $G_1^*(s)$ and $G_2^*(s)$ as increasing functions of $G_3^*(s)$. Then, if we replace $G_3^*(s)$ with a higher value 1, $G_1^*(s)$ and $G_2^*(s)$ implicitly defined in (12) and (13) should be lower. We can verify that (12) and (13) become (10) and (11) if we replace $G_3^*(s)$ with 1. Therefore, $\bar{G}_1(s) < G_1^*(s)$ and $\bar{G}_2(s) < G_2^*(s)$. ■

Proof of Proposition 1. We claim that in every equilibrium, given others' equilibrium strategies, player 3's expected value of prizes at $\bar{s}_3^*$ is x_3 as in Definition i). We prove the claim in two steps.

First, in every equilibrium, $\bar{s}_3^* = \inf \arg\max_{s \in [\underline{s}_1, \bar{s}_1]} U_3(s|\bar{G}_1, \bar{G}_2)$, where $\bar{G}_1(s)$ and $\bar{G}_2(s)$ are defined in Lemma 7 and $\underline{s}_1, \bar{s}_1$ solve $\bar{G}_1(\underline{s}_1) = 0$ and $\bar{G}_1(\bar{s}_1) = 1$. To see why, notice that $\bar{G}_i(s) = G_i^*(s)$ for any $s \geq \bar{s}_3^*$ and $i = 1, 2$, so the definition of equilibrium implies

$$u_3^* = U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2) \geq U_3(s|\bar{G}_1, \bar{G}_2) \tag{14}$$

for any $s \geq \bar{s}_3^*$. Therefore, $\inf \arg\max_{s \in [\underline{s}_1, \bar{s}_1]} U_3(s|\bar{G}_1, \bar{G}_2) \leq \bar{s}_3^*$. As a result, it remains to be shown that any $s \in [\underline{s}_1, \bar{s}_3^*)$ does not maximize $U_3(s|\bar{G}_1, \bar{G}_2)$.

Lemma 7 implies that $\bar{G}_1$ reaches zero at a higher performance level than G_1^* does; that is, $\underline{s}_1^* < \underline{s}_1$. The lemma also implies $U_3(s|\bar{G}_1, \bar{G}_2) < U_3(s|G_1^*, G_2^*) \leq u_3^*$ for $s \in (\underline{s}_1^*, \bar{s}_3^*)$. Because $[\underline{s}_1, \bar{s}_3^*)$ is a subset of $(\underline{s}_1^*, \bar{s}_3^*)$, we have $U_3(s|\bar{G}_1, \bar{G}_2) < u_3^*$ for $s \in [\underline{s}_1, \bar{s}_3^*)$. Recall that $u_3^* = U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2)$ in (14), so $U_3(s|\bar{G}_1, \bar{G}_2) < U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2)$ for $s \in [\underline{s}_1, \bar{s}_3^*)$. Hence, any $s \in [\underline{s}_1, \bar{s}_3^*)$ does not maximize $U_3(s|\bar{G}_1, \bar{G}_2)$.

Second, given others' equilibrium strategies, player 3's expected prize at $\bar{s}_3^*$ equals x_3 as in Definition i). To see why, notice that Lemma 5, the "Stochastic Dominance" property, implies $\bar{s}_i^* \geq \bar{s}_3^*$ for $i = 1, 2$. In addition, we must have $\bar{s}_1^* = \bar{s}_2^*$ because of Lemma 1, the property of "No Aggregate Gaps". Therefore, $u_i^* = v_1 - c_i\bar{s}_1^*$ for $i = 1, 2$. Substituting them into (10) and

(11), we can verify that

$$G_i^2(s) = \bar{G}_i(s + \bar{s}_1^* - \bar{s}_1^2) \tag{15}$$

for $i = 1, 2$. That is, $\bar{G}_i$ is G_i^2 shifted horizontally by $\bar{s}_1^* - \bar{s}_1^2$ for $i = 1, 2$. Therefore, the maximizer $\bar{s}_3^* = \inf \arg\max_{s \in [\underline{s}_1, \bar{s}_1]} U_3(s|\bar{G}_1, \bar{G}_2)$ is the maximizer $\hat{s}_3 = \inf \arg\max_{s \in [0, \bar{s}_1^2]} U_3(s|G_1^2, G_2^2)$ with the same shift. That is,

$$\hat{s}_3 = \bar{s}_3^* - (\bar{s}_1^* - \hat{s}_1) \tag{16}$$

Substituting (15) and (16) into the definition $x_3 = U_3(\hat{s}_3|G_1^2, G_2^2) + c_3\hat{s}_3$, we have $x_3 = U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2) + c_3\bar{s}_3^*$. Recall that $\bar{G}_i(s) = G_i^*(s)$ for $s \geq \bar{s}_3^*$, so $x_3 = U_3(\bar{s}_3^*|G_1^*, G_2^*) + c_3\bar{s}_3^*$, which is exactly player 3's expected value of prizes at $\bar{s}_3^*$ given others' equilibrium strategies. Hence, we prove the claim.

Therefore, $T = \bar{s}_3^*$ and $w_3 = 0$. Recall that Lemma 6 implies $u_3^* = 0$, so $u_3^* = \max(0, w_3)$. It remains to be shown that $u_i^* = \max(0, w_i)$ for $i = 1, 2$. As in the second step above, we can show that $x_i = v_1 G_j^*(\bar{s}_3^*) + v_2(1 - G_j^*(\bar{s}_3^*)) - c_i\bar{s}_3^*$ for $i, j \in \{1, 2\}$ and $i \neq j$. The definition of w_i implies $w_i = x_i - c_iT = v_1 G_j^*(\bar{s}_3^*) + v_2(1 - G_j^*(\bar{s}_3^*)) - c_i\bar{s}_3^* = u_i^*$ for distinct $i, j \in \{1, 2\}$. ■

Having proved Proposition 1, we can use it and Definitions i) to iii) to derive the closed-form expressions of equilibrium payoffs in Proposition 2.

Proof of Proposition 2. We have shown $u_3^* = 0$, so it remains to derive the payoffs for 1 and 2. Consider Case II first. Recall that $U_3(\cdot|G_1^2, G_2^2)$ is maximized at the upper boundary $\bar{s}_1^2$ if (6) does not hold but (7) does. Therefore, Definitions i) to iii) imply $x_i = v_1$ for $i = 1, 2, 3$ and $r_3 = v_1/c_3$, so $u_i^* = v_1(1 - c_i/c_3)$ for $i = 1, 2$.

Consider Case I. Recall that the lower boundary 0 is a maximizer of $U_3(\cdot|G_1^2, G_2^2)$ over $[0, \bar{s}_1^2]$ if neither (6) nor (7) holds. Therefore, Definitions i) to iii) imply $x_1 = v_1 - (v_1 - v_2)c_1/c_2$, $x_2 = v_2$ and $x_3 = v_2(1 - c_1/c_2)$. In addition, $x_3 - c_3T = 0$, so $T = v_2(1 - c_1/c_2)/c_3$. The definitions also imply $u_i^* = x_i - c_iT$ for $i = 1, 2$. Substituting x_1, x_2 and T into this expression, we obtain the expressions of u_1^* and u_2^* in Case I.

Consider Case III. Recall that (6) implies $U_3(\cdot|G_1^2, G_2^2)$ has an interior maximizer in $[0, \bar{s}_1^2]$. Rearranging terms, we have

$$U_3(s|G_1^2, G_2^2) = \frac{\Delta_1 - \Delta_2}{\Delta_1^2} c_1c_2s^2 + \left[\frac{\Delta_1 - \Delta_2}{\Delta_1}(c_2 - c_1) + \frac{\Delta_2}{\Delta_1}(c_1 + c_2) - c_3\right] s + \Delta_2\left(1 - \frac{c_1}{c_2}\right)$$

which is a quadratic function of s. The maximum of the quadratic function is

$$\hat{u}_3 = \Delta_2\left(1 - \frac{c_1}{c_2}\right) - \frac{c_1c_2}{4}(\Delta_1 - \Delta_2)\left[\frac{c_2 - c_1}{c_1c_2} + \frac{c_1 + c_2}{c_1c_2}\frac{\Delta_2}{\Delta_1 - \Delta_2} - \frac{\Delta_1}{\Delta_1 - \Delta_2}\frac{c_3}{c_1c_2}\right]^2 \tag{17}$$

Proposition 1 implies $x_3 - c_3T = u_3^* = 0$, and the definition of x_3 implies $x_3 - c_3\hat{s}_3 = \hat{u}_3$. Therefore,

$$T - \hat{s}_3 = \hat{u}_3/c_3 \tag{18}$$

Then, for $i = 1, 2$, we have $u_i^* = x_i - c_i T = u_i^2 + c_i \hat{s}_3 - c_i T = u_i^2 - \hat{u}_3 c_i / c_3$, where the first equality is from Proposition 1, the second from the definition of x_i and the last from (18). Substituting $u_i^2 = v_1 - \Delta_1 c_i / c_2$ into the above expression of u_i^*, we obtain the payoff expressions in Case III. ■

Proof of Corollary 1. We can verify that the equilibrium payoffs in Proposition 2 are continuous in Δ_1/Δ_2, so it is sufficient to prove the corollary in each of Case I, II, and III. Consider Case II first. Using Proposition 2, we can verify that $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ is independent of $\mathbf{v}$, so it is nondecreasing in Δ_1/Δ_2.

Consider Case I. Substituting the payoff expressions in Proposition 2 into $(u_1^* - u_2^*)/(u_2^* - u_3^*)$, we can rewrite it as

$$\frac{u_1^* - u_2^*}{u_2^* - u_3^*} = \left[\frac{\Delta_1}{\Delta_2}\left(1 - \frac{c_1}{c_2}\right) + 1 - \frac{c_1}{c_3}\left(1 - \frac{c_1}{c_2}\right)\right] \Big/ \left[1 - \frac{c_2}{c_3}\left(1 - \frac{c_1}{c_2}\right)\right]$$

which is increasing in Δ_1/Δ_2.

Consider Case III. In the proof of Proposition 1, we show that in any equilibrium, $u_i^* = v_1 - c_i \bar{s}_1^*$ for $i = 1, 2$. Then, using the expressions of u_1^*, u_2^* and $u_3^* = 0$, we can rewrite

$$\frac{u_1^* - u_2^*}{u_2^* - u_3^*} = \frac{v_1/v_2 - c_1 \bar{s}_1^*/v_2}{v_1/v_2 - c_2 \bar{s}_1^*/v_2} - 1$$

Notice that $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ being nondecreasing in Δ_1/Δ_2 is equivalent to $v_1/v_2 - c_1 \bar{s}_1^*/v_2 - (v_1/v_2 - c_2 \bar{s}_1^*/v_2) = (c_2 - c_1)\,\bar{s}_1^*/v_2$ being nondecreasing in Δ_1/Δ_2. Hence, it is sufficient to show that $\bar{s}_1^*/v_2$ is nondecreasing in Δ_1/Δ_2.

Recall that $u_1^* = v_1 - c_1 \bar{s}_1^*$, so $\bar{s}_1^*/v_2 = (v_1 - u_1^*)/(c_3 v_2)$. In addition, because of the expression of u_1^* in Proposition 2 and (17), we can rewrite $\bar{s}_1^*/v_2$ as

$$\frac{\bar{s}_1^*}{v_2} = \frac{1}{c_3}\left(1 - \frac{c_1}{c_2}\right) + \frac{\Delta_1}{\Delta_2}\frac{1}{c_2} + \frac{1}{4c_1c_2c_3}\frac{\left[(c_3 - c_2 - c_1) - (1 - \Delta_1/\Delta_2)\,(c_3 + c_1 - c_2)\right]^2}{1 - \Delta_1/\Delta_2}$$

Consider the last term. Its denominator is decreasing in Δ_1/Δ_2. Its numerator is increasing in Δ_1/Δ_2 because $c_3 - c_2 - c_1 > 0$ and $c_3 + c_1 - c_2 > 0$ in Case III. Therefore, the last term is increasing in Δ_1/Δ_2, and hence, so is $\bar{s}_1^*/v_2$. ■

Next, we prove Propositions 3 to 6.

Proof of Proposition 3. We first use the algorithm to derive a set of strategies, then show that it is the unique equilibrium. Following the algorithm, $\mathcal{A}^+(0) = \{2, 3\}$, then we can extend G_2, G_3 by solving $v_2 G_i(s) - c_j s = u_j^*$ for $i, j \in \{2, 3\}$ and $j \neq i$. Notice that (6) and (7) imply that u_2^*, u_3^* are as in Case I in Proposition 2. Substituting the payoffs into the two equations above, we have

$$\begin{aligned} G_2(s) &= s c_3 / v_2 \\ G_3(s) &= s c_2 / v_2 + (c_3 - c_2 + c_1)/c_3 \end{aligned}$$

for $s \in [0, s']$ where $s' = (c_2 - c_1)v_2/(c_2c_3)$ solves $G_3(s') = 1$. We can verify that the first switch point above 0 is s', and $\mathcal{A}^+(s') = \{1, 2\}$, so we can extend G_1, G_2 by solving

$$v_1G_2(s) + v_2(1 - G_2(s)) - c_1s = u_1^* \tag{19}$$

$$v_1G_1(s) + v_2(1 - G_1(s)) - c_2s = u_2^* \tag{20}$$

Substituting expressions of u_1^* and u_2^* in Case I, we can solve the above equations and get

$$\begin{aligned} G_2(s) &= \frac{c_1}{\Delta_1}s + 1 - \frac{c_1}{c_2} - \frac{c_1}{c_3}\left(1 - \frac{c_1}{c_2}\right)\frac{\Delta_2}{\Delta_1} \\ G_1(s) &= \frac{c_2}{\Delta_1}s - \frac{\Delta_2}{\Delta_1}\frac{c_2 - c_1}{c_3} \end{aligned}$$

for $s \in [s', s'']$ where $s'' = \Delta_1/c_2 + \Delta_2(c_2 - c_1)/(c_2c_3)$ solves $G_1(s'') = 1$.

It is straightforward to verify that (G_1, G_2, G_3) is indeed an equilibrium, and we show below that there are no other equilibria. First, in any equilibrium, G_i^* for $i = 2, 3$ must satisfy $v_2G_i(s) - c_js = u_j^*$ for $j \in \{2, 3\}\backslash\{i\}$ and $s \in [0, \underline{s}_1^*]$, where $\underline{s}_1^*$ is the lower boundary of G_i^*'s support. Therefore, $G_i^*(s) = G_i(s)$ for $i = 2, 3$ and for $s \in [0, \underline{s}_1^*]$. At $\underline{s}_1^*$, we have $U_1(\underline{s}_1^*|G_2^*, G_3^*) = u_1^*$. The definition of switch point s' implies that it is the lowest performance level satisfying this property, so $\underline{s}_1^* = s'$. Notice that $G_3^*(s') = 1$, so $\underline{s}_1^* = \bar{s}_3^* = s'$. Hence, G_i^* for $i = 1, 2$ satisfies (19) and (20) for $s > s'$. Therefore, $G_i^*(s) = G_i(s)$ for $i = 1, 2$ and for $s \in [s', s'']$. As above, from the equilibrium payoffs, we uniquely determine the strategies in any equilibrium. Therefore, there are no other equilibria. ■

Proof of Proposition 4. As in Proposition 3, we first use the algorithm to construct a set of strategies. With $\mathcal{A}^+(0) = \{2, 3\}$, we can extend G_2 and G_3 by solving $v_2G_i(s) - c_js = u_j^*$ for $i, j \in \{2, 3\}$ and $j \neq i$. That is, $G_2(s) = sc_3/v_2$ and $G_3(s) = sc_2/v_2 + (1 - c_2/c_3)\,v_1/v_2$.

Denote the next switch point as s'. By its definition, s' is the smallest performance level such that $U_1(s|G_2, G_3) = u_1^*$. Therefore, we can extend G_1, G_2, G_3 by solving $U_i(s|G_j, G_k) = u_i^*$ for $i = 1, 2, 3$. If $\Delta_1/\Delta_2 = 1$, we can solve the linear equation system and obtain $G_i(s) = \sum_{j\neq i}(u_j^* + c_js)/(2v_2)$ for $i = 1, 2, 3$. If $\Delta_1/\Delta_2 \neq 1$, we can use the definition of A_i to rewrite $U_i(s|G_j, G_k) = u_i^*$ as

$$\left(G_j(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right)\left(G_k(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right) = \frac{A_i + c_is}{\Delta_1 - \Delta_2} \tag{21}$$

The product of (21) for $i = 1, 2, 3$ is

$$\left[\times_{i=1}^{3}\left(G_i(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right)\right]^2 = \times_{i=1}^{3}\left(\frac{A_i + c_is}{\Delta_1 - \Delta_2}\right)$$

therefore

$$\times_{i=1}^{3}\left(G_i(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right) = \pm\sqrt{\times_{i=1}^{3}\left(\frac{A_i + c_is}{\Delta_1 - \Delta_2}\right)} \tag{22}$$

Combining (21) and (22), we obtain

$$G_i(s) = \pm\frac{\Delta_1-\Delta_2}{A_i+c_is}\sqrt{\times_{i=1}^{3}\left(\frac{A_i+c_is}{\Delta_1-\Delta_2}\right)}-\frac{\Delta_2}{\Delta_1-\Delta_2}$$

On the one hand, if $\Delta_1-\Delta_2>0$, we can verify that $A_1+c_1s>A_2+c_2s>A_3+c_3s>0$ for $s\in(0,v_1/c_3)$, so

$$G_i(s) = \frac{1}{A_i+c_is}\sqrt{\frac{\times_{i=1}^{3}(A_i+c_is)}{\Delta_1-\Delta_2}}-\frac{\Delta_2}{\Delta_1-\Delta_2} \tag{23}$$

otherwise $G_i(s)<0$. Notice that A_i+c_is and $\Delta_1-\Delta_2$ are both positive, so the square root in (23) is a real number. Therefore, $F_i(s)$, which is the right hand side of (23), is also a real number. On the other hand, if $\Delta_1-\Delta_2<0$, we can verify that $A_3+c_3s<A_2+c_2s<A_1+c_1s<0$ for $s\in(0,v_1/c_3)$. Moreover, the second term in (23) satisfies $-\Delta_2/(\Delta_1-\Delta_2)>1$. Then, (23) is also true, otherwise $G_i(s)>1$. Notice that A_i+c_is and $\Delta_1-\Delta_2$ are negative, so the square root in (23) is a real number, so is $F_i(s)$. Hence, the strategies in Proposition 4 are the unique outcome of the algorithm.

Let us verify that G_i is nondecreasing for $i=1,2,3$. By the construction in the algorithm, G_i is also continuous. It is straightforward to verify that the linear parts of G_i are nondecreasing. Therefore, it remains to verify that G_i is nondecreasing over $s\in(\underline{s}_1^*,v_1/c_3)$. First, consider the case with $\Delta_1-\Delta_2>0$. Recall that in this case, $A_1+c_1s>A_2+c_2s>A_3+c_3s>0$ for $s\in(0,v_1/c_3)$, so

$$G_3(s) = \sqrt{\frac{(A_1+c_1s)(A_2+c_2s)}{(A_3+c_3s)(\Delta_1-\Delta_2)}}-\frac{\Delta_2}{\Delta_1-\Delta_2}$$

When $\Delta_1-\Delta_2>0$, $G_3(s)$ is a monotone transformation of $(A_1+c_1s)(A_2+c_2s)/(A_3+c_3s)$. Therefore, (9) implies the right derivative $G_3'(\underline{s}_1^*)\geq 0$.[21] In addition,

$$\frac{\partial}{\partial s}\left[\frac{(A_1+c_1s)(A_2+c_2s)}{A_3+c_3s}\right] = \frac{c_1c_2c_3(s^2+2\frac{A_3}{c_3}s)+c_1A_2A_3+A_1c_2A_3-A_1A_2c_3}{(A_3+c_3s)^2} \tag{24}$$

where the denominator is positive and the numerator is increasing in s. The positive denominator implies that $G_3'(s)$ and the numerator have the same sign. Recall that the numerator is increasing in s and it is nonnegative at $\underline{s}_1^*$ due to (9), so it is positive for $s>\underline{s}_1^*$. Therefore, $G_3'(s)>0$ for $s>\underline{s}_1^*$. Moreover, (8) can be rewritten as

$$((v_1-2v_2)G_3(s)+v_2)g_2(s)+((v_1-2v_2)G_2(s)+v_2)g_3(s) = c_1 \tag{25}$$

$$((v_1-2v_2)G_3(s)+v_2)g_1(s)+((v_1-2v_2)G_1(s)+v_2)g_3(s) = c_2 \tag{26}$$

$$((v_1-2v_2)G_2(s)+v_2)g_1(s)+((v_1-2v_2)G_1(s)+v_2)g_2(s) = c_3 \tag{27}$$

Recall that $A_1+c_1s>A_2+c_2s>A_3+c_3s>0$ for $s\in(0,v_1/c_3)$ if $\Delta_1-\Delta_2>0$. Therefore, (23) implies $G_1(s)<G_2(s)<G_3(s)$ for $s\in(0,v_1/c_3)$. Comparing (25) and (26), we obtain

[21] We use the right derivative because, for $s<\underline{s}_1^*$, the expression of $G_3(s)$ is different.

$g_2(s) < g_1(s)$. Similar, comparing (26) and (27), we obtain $g_3(s) < g_2(s)$. Therefore, $0 < G'_3(s) < G'_2(s) < G'_1(s)$.

We have verified that G_i is nondecreasing if $\Delta_1 - \Delta_2 > 0$. Next, we consider the case with $\Delta_1 - \Delta_2 < 0$. As above, we have $G'_3(s) < G'_2(s) < G'_1(s)$, so it remains to show $G'_3(s) > 0$ for $s \in (\underline{s}_1^*, v_1/c_3)$. Recall that, if $\Delta_1 - \Delta_2 < 0$, we have $A_3 + c_3 s < A_2 + c_2 s < A_1 + c_1 s < 0$ for $s \in (0, v_1/c_3)$, so

$$G_3(s) = -\sqrt{\frac{(A_1 + c_1 s)(A_2 + c_2 s)}{(A_3 + c_3 s)(\Delta_1 - \Delta_2)}} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$$

Using the same argument in the case of $\Delta_1 - \Delta_2 > 0$, we obtain that $G'_3(s)$ and the numerator in (24) also have the same sign for $s \in (\underline{s}_1^*, v_1/c_3)$. Notice that the numerator is a quadratic function that reaches its minimum at $s = -A_3/c_3 = (\Delta_2)^2/((\Delta_2 - \Delta_1)c_3) > v_1/c_3$, so it is decreasing in s for $s \in (\underline{s}_1^*, v_1/c_3)$. Notice that $G_i(v_1/c_3) = 1$, so (25)-(27) imply the left derivative $G'_3(v_1/c_3) = (c_1 + c_2 - c_3)/(2\Delta_1) \geq 0$, where the inequality is from the assumption $c_1 + c_2 \geq c_3$ of the proposition. As a result, the numerator in (24) is nonnegative at $s = v_1/c_3$, so it is positive over for $s \in (\underline{s}_1^*, v_1/c_3)$ because it is decreasing in s. Hence, (24) is positive and G_3 is increasing.

Now we verify that the constructed strategies are a unique equilibrium. As in Proposition 3, $G_i^*(s) = G_i(s)$ for $s \leq s'$. We show $G_i(s) = G_i^*(s)$ for $i = 1, 2, 3$ and for $s > s'$ in two steps.

First, $\underline{s}_1^* = s'$. As in Proposition 3, the definition of s' implies that $\underline{s}_1^* \geq s'$. Suppose $\underline{s}_1^* > s'$, then $G_2^*(s), G_3^*(s)$ satisfy $v_2 G_2^*(s) - c_3 s = u_3^*$ and $v_2 G_3^*(s) - c_2 s = u_2^*$ for $s \in (s', s'')$. Following the same argument in the proof of Lemma 7, we can verify that $G_2^*(s) > G_2(s)$ and $G_3^*(s) > G_3(s)$ for $s \in (s', s'')$. Therefore, $U_1(s|G_2^*, G_3^*) > U_1(s|G_2, G_3) = u_1^*$ for $s \in (s', s'')$, where the equality is from the definition of G_2, G_3. Hence, $\underline{s}_1^* = s'$.

Second, any $s \in [s', s'']$ is in the support of G_i^* for $i = 1, 2, 3$. Suppose otherwise that there exists $\varepsilon > 0$ and $s_3 \in [s', s'']$ such that $(s_3, s_3 + \varepsilon)$ is not a subset of G_i^*'s support. Then, Lemma 4, the property of "Nested Gaps", implies that G_i^* must be G_3^*. Without loss of generality, assume s_3 is the smallest performance level with the above property. Then, $G_3^*(s) = G_3^*(s_3)$ for $s \in [s_3, s_3 + \varepsilon]$. Moreover, $G_3(s) > G_3^*(s)$ because G_3 is increasing and $G_3(s_3) = G_3^*(s)$. Then, as in the proof of Lemma 7, we have $G_i(s) < G_i^*(s)$ for $i = 1, 2$ and $s \in [s_3, s_3 + \varepsilon]$. Therefore, $U_3(s|G_1^*, G_2^*) > U_3(s|G_1, G_2) = u_3^*$ for $s \in [s_3, s_3 + \varepsilon]$. This is a contradiction.

In these two steps, we verify that G_i^* and G_i have the same support for $i = 1, 2, 3$. Moreover, the construction above shows that, given the equilibrium payoffs, G_i is the unique strategy with such support. Hence, there are no other equilibria. ■

Proof of Proposition 5. The algorithm implies that there are two switch points between 0 and $(v_1 - u_1^*)/c_1$, where the algorithm ends. Moreover, they satisfy $s' < s''$, and $\mathcal{A}^+(s') = \{1, 2, 3\}$ and $\mathcal{A}^+(s'') = \{1, 2\}$. Following the calculations in the proof of Proposition 4, we can construct the strategies in this proposition.

Let us verify that the constructed G_i are indeed non-decreasing. We first verify that the left derivative $G'_3(\bar{s}_3^*) = 0$. Notice that this proposition corresponds to Case III in Proposition

2, so $U_3(\cdot|G_1^2, G_2^2)$ has an interior maximizer over $[0, \bar{s}_1^2]$. In addition, according to the proof of Proposition 1, $U_3(\cdot|G_1^2, G_2^2)$ is $U_3(\cdot|\bar{G}_1, \bar{G}_2)$ with a horizontal shift. Therefore, $U_3(\cdot|\bar{G}_1, \bar{G}_2)$ also has an interior maximizer over $[\underline{s}_1, \bar{s}_1]$, and the maximizer is $\bar{s}_3^*$. The corresponding first order condition is (27) with $s = \bar{s}_3^*$. We can also verify that $g_1(\bar{s}_3^*) = c_2/\Delta_1$, $g_2(\bar{s}_3^*) = c_1/\Delta_1$ and $g_3(\bar{s}_3^*) = 0$ solve (25)-(27), so the left derivative $G_3'(\bar{s}_3^*) = g_3(\bar{s}_3^*) = 0$.

Second, G_3 is nondecreasing over $(\underline{s}_1^*, \bar{s}_3^*)$. Recall that for $s \in (\underline{s}_1^*, \bar{s}_3^*)$, $G_3'(s)$ and the numerator in (24) have the same sign. Notice that $G_3(\bar{s}_3^*) = 1$, so $G_3(s) < 1$ implies $G_3'(s) \geq 0$ for s slightly below $\bar{s}_3^*$. Therefore, the numerator is also nonnegative at s slightly below $\bar{s}_3^*$. In addition, the numerator is zero at $s = \bar{s}_3^*$ due to the first step. Therefore, the numerator, which is a U-shaped quadratic function of s, is positive for $s < \bar{s}_3^*$. Hence, $G_3'(s) > 0$ for $s \in (\underline{s}_1^*, \bar{s}_3^*)$.

As in the proof of Proposition 4, $G_1'(s) > G_2'(s) > G_3'(s)$ for $s \in (\underline{s}_1^*, \bar{s}_3^*)$, so G_i is increasing over $(\underline{s}_1^*, \bar{s}_3^*)$. The other parts of the strategies are linear, and are also non-decreasing.

By the two steps in the proof of Proposition 4, we can prove the constructed strategies are a unique equilibrium. ■

Proof of Proposition 6. The algorithm implies that there are two switch points between 0 and $(v_1 - u_1^*)/c_1$. Moreover, $s' < s''$, and $\mathcal{A}^+(s') = \{1, 2\}$ and $\mathcal{A}^+(s'') = \{1, 2, 3\}$. As in the proof of Proposition 4, we have $G_i^*(s) = G_i(s)$ for $s \leq s'$. At the first switch point s', we have $g_3(s') < 0$, so at least one of G_1^*, G_2^*, G_3^*'s supports does not contain $(s', s' + \varepsilon)$ for some $\varepsilon > 0$. Then, the property of "Nested Gaps" in Lemma 4 implies that $(s', s' + \varepsilon)$ is not a subset of G_3^*'s support. By the definition of the second switch point s'', (s', s'') cannot be in the support of G_3^*. By the same argument for $[s', s'']$ in Proposition 4, we have $[s'', (v_1 - u_1^*)/c_1]$ is in the support of G_i^* for $i = 1, 2, 3$.

Let us verify that G_i is nondecreasing. As above, it is sufficient to verify that G_3 is nondecreasing for its nonlinear part over $(\hat{s}_3, \bar{s}_1^*)$. As in the proof of Proposition 4, $\Delta_1 - \Delta_2 > 0$ implies that the numerator in (24) is increasing over $(0, \bar{s}_1^*)$. Because $F_3'(s)$ has the same sign as the numerator, so $F_3(s)$ is a U-shaped function over $(0, \bar{s}_1^*)$. By their definitions, $F_i(\underline{s}_1^*)$ for $i = 1, 2, 3$ are the unique solution in $[0, 1]^3$ to the system $U_1(\underline{s}_1^*|F_2, F_3) = u_1^*$, $U_2(\underline{s}_1^*|F_1, F_3) = u_2^*$ and $U_3(\underline{s}_1^*|F_1, F_2) = u_3^*$. In addition, $U_1(\underline{s}_1^*|G_2, G_3) = u_1^*$, $U_2(\underline{s}_1^*|G_1, G_3) = u_2^*$ and $U_3(\underline{s}_1^*|G_1, G_2) = u_3^*$. Therefore, $G_i(\underline{s}_1^*) = F_i(\underline{s}_1^*)$. The construction of G_3 implies $G_3(\underline{s}_1^*) = G_3(\hat{s}_3) = F_3(\hat{s}_3)$, so $F_3(\underline{s}_1^*) = F_3(\hat{s}_3)$. Hence, the U-shaped function F_3 is increasing over $(\hat{s}_3, \bar{s}_1^*)$.

The first paragraph shows that we uniquely determine the supports of strategies in any equilibrium. Therefore, the construction implies that given the equilibrium payoffs, there are no other strategies with the same supports. Hence, there are no other equilibria. ■

We first introduce a lemma below, then use it to prove Proposition 7.

Lemma 8 *If $1 < \Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and $c_3 < c_1 + c_2$, there exists a unique $\lambda \in (1, c_1/(c_3 - c_2))$ such that $F_3'(\underline{s}_1^*) > 0$ if and only if $\Delta_1/\Delta_2 \in (1, \lambda)$.*

Proof. We prove this in three steps. First, $\underline{s}_1^*/v_1$ increases in κ, where $\kappa \equiv 1/((\Delta_1/\Delta_2)^2 - 1)$.

To see why, recall that $\underline{s}_1^*$ solves $G_1^*(\underline{s}_1^*) = 0$. That is,

$$\frac{1}{c_1\underline{s}_1^* + A_1}\sqrt{\frac{\times_{i=1}^3 (c_i\underline{s}_1^* + A_i)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2} = 0$$

Because $\Delta_1 - \Delta_2 > 0$, we can rewrite it as $(c_2\underline{s}_1^* + A_2)(c_3\underline{s}_1^* + A_3)/(c_1\underline{s}_1^* + A_1) = \Delta_2^2/(\Delta_1 - \Delta_2)$, or

$$c_2c_3\underline{s}_1^{*2} + \left(A_2c_3 + c_2A_3 - \frac{\Delta_2^2}{\Delta_1 - \Delta_2}c_1\right)\underline{s}_1^* + A_2A_3 - \frac{\Delta_2^2}{\Delta_1 - \Delta_2}A_1 = 0 \tag{28}$$

Recall that $A_i = v_1(1 - c_i/c_3) + \Delta_2^2/(\Delta_1 - \Delta_2)$, so

$$\frac{A_i}{v_1} = 1 - \frac{c_i}{c_3} + \frac{\Delta_2^2}{(\Delta_1 - \Delta_2)(\Delta_1 + \Delta_2)} = 1 - \frac{c_i}{c_3} + \frac{1}{t^2 - 1} \tag{29}$$

$$\frac{\Delta_2^2}{(\Delta_1 - \Delta_2)v_1} = \frac{1}{t^2 - 1} \tag{30}$$

where $t \equiv \Delta_1/\Delta_2$. Dividing both sides of (28) by v_1^2 and using (29) and (30), we can rewrite (28) as

$$c_2c_3\left(\frac{\underline{s}_1^*}{v_1}\right)^2 + [c_3 - c_2 + \kappa(c_3 + c_2 - c_1)]\frac{\underline{s}_1^*}{v_1} - \kappa\frac{c_2 - c_1}{c_3} = 0 \tag{31}$$

where $\kappa = 1/(t^2 - 1)$. Because the last term is negative, the above equation has two roots of different signs. Therefore, the positive root is

$$\frac{\underline{s}_1^*}{v_1} = \frac{-(c_3 - c_2 + \kappa(c_3 + c_2 - c_1)) + \sqrt{(c_3 - c_2 + \kappa(c_3 + c_2 - c_1))^2 + 4c_2(c_2 - c_1)\kappa}}{c_2c_3} \tag{32}$$

Recall that $F_3(s) = \frac{1}{A_3 + c_3s}\sqrt{\frac{\times_{i=1}^3(c_is + A_i)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$. Because $\Delta_1 - \Delta_2 > 0$, $F_3'(\underline{s}_1^*) > 0$ is equivalent to (9). Expanding the derivative in (9), we obtain

$$c_1(A_2 + c_2\underline{s}_1^*)(A_3 + c_3\underline{s}_1^*) + (A_1 + c_1\underline{s}_1^*)c_2(A_3 + c_3\underline{s}_1^*) - (A_1 + c_1\underline{s}_1^*)(A_2 + c_2\underline{s}_1^*)c_3 > 0$$

Dividing both sides by $c_1c_2c_3\underline{s}_1^*$, we obtain

$$\left(\frac{A_2}{\underline{s}_1^*c_2} + 1\right)\left(\frac{A_3}{\underline{s}_1^*c_3} + 1\right) + \left(\frac{A_1}{\underline{s}_1^*c_1} + 1\right)\left(\frac{A_3}{\underline{s}_1^*c_3} + 1\right) - \left(\frac{A_1}{\underline{s}_1^*c_1} + 1\right)\left(\frac{A_2}{\underline{s}_1^*c_2} + 1\right) > 0$$

Notice that $\frac{A_i}{\underline{s}_1^*c_i} = \frac{1 - c_i/c_3 + \kappa}{c_i}\frac{1}{\underline{s}_1^*/v_1}$, so the inequality above is

$$\left(\frac{1 - c_2/c_3 + \kappa}{c_2} + \frac{\underline{s}_1^*}{v_1}\right)\left(\frac{1 - c_3/c_3 + \kappa}{c_3} + \frac{\underline{s}_1^*}{v_1}\right) + \left(\frac{1 - c_1/c_3 + \kappa}{c_1} + \frac{\underline{s}_1^*}{v_1}\right)\left(\frac{1 - c_3/c_3 + \kappa}{c_3} + \frac{\underline{s}_1^*}{v_1}\right)$$
$$- \left(\frac{1 - c_1/c_3 + \kappa}{c_1} + \frac{\underline{s}_1^*}{v_1}\right)\left(\frac{1 - c_2/c_3 + \kappa}{c_2} + \frac{\underline{s}_1^*}{v_1}\right) > 0$$

Collecting terms with respect to $\underline{s}_1^*/v_1$, we get the left-hand side of the above inequality

$$LHS = \left(\frac{\underline{s}_1^*}{v_1}\right)^2 + 4\frac{\kappa}{c_3}\frac{\underline{s}_1^*}{v_1} + \frac{1 - c_1/c_3 + \kappa}{c_1} + \frac{1 - c_2/c_3 + \kappa}{c_2} + \frac{1 - c_3/c_3 + \kappa}{c_3} \tag{33}$$

Recall that $\underline{s}_1^*/v_1$ solves (31), which can be rewritten as

$$c_2 c_3 \left(\frac{\underline{s}_1^*}{v_1} + \frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{2c_1 c_2}\right)^2 - \left[\left(\frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{2c_1 c_2}\right)^2 + \kappa\frac{c_2 - c_1}{c_3}\right] = 0$$

If κ increases, $\frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{2c_1 c_2}$ increases and the last term decreases. Then, the quadratic function of $\underline{s}_1^*/v_1$ shifts to the right and downwards. This implies that the larger root $\underline{s}_1^*/v_1$ increases.

Second, LHS in (33) is increasing in κ for $\kappa \in [\underline{\kappa}, +\infty)$ if it is increasing at $\underline{\kappa}$, where

$$\underline{\kappa} = \left.\frac{1}{t^2 - 1}\right|_{t = c_1/(c_3 - c_2)} = \frac{(c_3 - c_2)^2}{(c_1 + c_2 - c_3)(c_1 + c_3 - c_2)}$$

The first step implies that $\underline{s}_1^*/v_1$ increases in κ, so the first two terms in (33) increase in κ. However, the last three terms do not. To see why, the derivative of the last three terms with respect to κ is

$$2\kappa\frac{c_1 + c_2 - c_3}{c_1 c_2 c_3} - \frac{2(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}$$

which is increasing in κ. However, the derivative at $\kappa = \underline{\kappa}$ is

$$-\frac{2c_1(c_2 - c_1)(c_3 - c_2)}{c_1 c_2 c_3^2 (c_1 + c_3 - c_2)} < 0 \tag{34}$$

Therefore, the sum of the last three terms is a U-shaped quadratic function of κ. Hence, to show (33) is increasing in κ, it is sufficient to show it is increasing in κ at $\underline{\kappa}$.

Third, LHS in (33) is increasing in κ at $\underline{\kappa}$. Substituting (34) into the derivative of (33) with respect to κ at $\underline{\kappa}$, we have

$$\left.\frac{\partial LHS}{\partial \kappa}\right|_{\underline{\kappa}} = \left(2y + \frac{4}{c_3}\kappa\right)\left.\frac{\partial y}{\partial \kappa}\right|_{\underline{\kappa}} + \frac{4}{c_3}y - \frac{2c_1(c_2 - c_1)(c_3 - c_2)}{c_1 c_2 c_3^2 (c_1 + c_3 - c_2)} \tag{35}$$

where $y \equiv \underline{s}_1^*/v_1$. Recall that if κ increases, the quadratic function in (31) shifts to the right and downwards. Therefore, the larger root y of the equation shifts to the right by at least $\frac{c_3 + c_2 - c_1}{2c_2 c_3}$. That is, $\left.\frac{\partial y}{\partial \kappa}\right|_{\underline{\kappa}} \geq \frac{c_3 + c_2 - c_1}{2c_2 c_3}$. Substituting $\underline{\kappa}$ into (32), we get

$$y(\underline{\kappa}) = \frac{(c_2 - c_1)(c_3 - c_2)}{c_2 c_3 (c_3 - c_2 + c_1)}$$

Substituting $y(\underline{\kappa})$, $\underline{\kappa}$ and the lower bound of $\partial y(\underline{\kappa})/\partial \kappa$ into (35), then multiplying both sides by

$c_2c_3^2/2$, we get

$$\frac{\partial LHS}{\partial \kappa}\bigg|_{\underline{\kappa}} \frac{c_2c_3^2}{2} \quad = \quad \left(\frac{c_3+c_2-c_1}{2c_2}+1\right)\frac{(c_2-c_1)(c_3-c_2)}{c_1+c_3-c_2}+\frac{2\sum_{i=1}^{3}(c_1+c_2+c_3-2c_i)}{(c_3-c_2)^2}>0$$

Therefore, (33) is increasing in κ. Hence, there exists λ such that the lemma holds. ■

Proof of Proposition 7. Consider $c_3 > c_1 + c_2$ first. Then, $c_1/(c_3 - c_2) < 1$ and $2c_1/(c_3 - c_2 + c_1) < 1$. Therefore, Proposition 3 implies that the equilibrium is of Type I for $\Delta_1/\Delta_2 \geq 2c_1/(c_3 - c_2 + c_1)$. If $\Delta_1/\Delta_2 \in (0, 2c_1/(c_3 - c_2 + c_1))$, Proposition 4 implies that the equilibrium is of Type III. If $\Delta_1/\Delta_2 = 0$, $v_1 = v_2 > 0$, so the equilibrium is of Type II.

Consider $c_3 \leq c_1 + c_2$. First, Proposition 4 implies that the equilibrium is of Type II for $\Delta_1/\Delta_2 \leq 1$. Second, Proposition 3 implies that the equilibrium is of Type I for $\Delta_1/\Delta_2 \geq c_1/(c_3 - c_2)$. Third, according to Proposition 4, the equilibrium is of Type II if $1 < \Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and if $F_3'(\underline{s}_1^*) \geq 0$. Lemma 8 implies that the equilibrium is of Type II for $\Delta_1/\Delta_2 \in (1, \lambda]$. Then, the first step implies that the equilibrium is of Type II for $\Delta_1/\Delta_2 \leq \lambda$. Fourth, if $\Delta_1/\Delta_2 \in (\lambda, c_1/(c_3 - c_2))$, Lemma 8 implies that $1 < \Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and $F_3'(\underline{s}_1^*) < 0$. Then, Proposition 6 implies that the equilibrium is of Type II$'$. ■

Proof of Proposition 8. Lemma 3 implies that players $4, ..., n$ choose $s = 0$ with probability 1. In addition, Lemma 2 implies that they do not win any prize in any equilibrium. Therefore, their equilibrium payoffs are zero.

Next, consider players 1,2 and 3. Because players $4, ..., n$ do not choose positive performance and do not win any prize, players $1, 2, 3$'s strategies in every equilibrium in the n-player contest are also an equilibrium in the three-player contest. Then, the unique equilibrium of the three-player contest implies that the n-player contest has a unique equilibrium, in which players $1, 2, 3$ use the same strategies as in the three-player contest. Moreover, their payoffs in the n-player contest are also the same as those in the three-player contest. ■