Ability Grouping in All-Pay Contests*

Jun Xiao†

October 11, 2017

Abstract

This paper considers a situation in which participants with heterogeneous ability types are grouped into different competitions for performance ranking. A planner can allocate both the participants and a fixed amount of prize money across all-pay contests in order to maximize a weighted sum of total performance subject to a restriction that no player chooses non-performance. The weights are type-specific. We show that, whatever the weights are, separating – assigning participants with the same ability together – is superior to mixing – assigning participants with different abilities together. Moreover, we also characterize the associated optimal prize structures.

JEL classification: D72, D44, L22

Keywords: all-pay, contest, asymmetric, mixing, tracking

*First draft: May 2013. We would like to thank Nisvan Erkal, Vijay Krishna, Simon Loertscher, Claudio Mezzetti, John Morgan, participants in conferences and seminars for comments and discussion, and David Marks for research assistance. Financial support from the University of Melbourne is greatly appreciated.

†Department of Economics, University of Melbourne. E-mail: jun.xiao@unimelb.edu.au.

1 Introduction

Consider a situation in which a school wants to assign a group of students to different classrooms. Should the school group students with similar abilities together – a practice known as "tracking"– so that high ability students are separated from low ability ones, or should the school have mixed classrooms in which students of different abilities are grouped together? Tracking was very common in US schools but became less popular in the late 1980s due to the criticism of trapping students of low socioeconomic status in low-level groups. However, tracking has returned to the attention of educators recently. According to Yee (2013), "... of the fourth-grade teachers surveyed, 71 percent said they had grouped students by reading ability in 2009, up from 28 percent in 1998". Different grouping policies can be observed not only over time but also across different countries. For instance, in Germany, pupils after primary schooling are grouped into three types of secondary schools to receive training for blue-collar apprenticeships, apprenticeship training in white-collar occupations, or training for further education. In contrast, tracking was explicitly discontinued in China in 2006.[1] Not surprisingly, tracking has also been a controversial topic in the economic literature on education, and there has been a long debate on this issue from many different perspectives: students' achievement, equity, and even morality.[2]

In this paper, we examine the competitive effects of tracking and ask whether or not it enhances students' expected performance when their grades or rewards depend on their relative performance.[3] Two features are important to a social planner. First, non-performance is not desirable. For instance, a school may want every student to graduate or at least attend the classes. This requirement is referred to as the positive performance requirement, and it imposes a challenge to go beyond well-studied contest forms. For instance, if the players are identical, an all-pay auction maximizes the total performance (Baye et al. 1996). However, the positive performance requirement is violated as there are equilibria in which some players exhibit non-performance.

Second, the planner may weigh students of different abilities differently. For example, a school may care more about high ability students' performance than others. As a result, it is important to examine how different weights affect the comparison of tracking and mixing.

Our results imply that, with a well-designed award system, tracking/separating – assigning students with the same ability together – is superior to mixing – assigning

[1] Policies that forbade tracking in schools started in the 1990s, and a national law was passed in 2006. In contrast, tracking remains common in Chinese universities.

[2] See, for instance, Loveless (2013).

[3] We do not consider the equity issue nor the effects of tracking on the quality of instruction. If students are tracked, the classes are more homogeneous and therefore they could be easier to teach. See, for instance, Duflo et al. (2011).

students with different abilities together. This result is true no matter how a planner weighs different ability groups.

Our model also applies to the quota systems in college admissions, in which minority students compete separately for the reserved admission quota.[4] Compared to majority students, it is usually more costly for the minority students to acquire the same level of academic achievement. Therefore, the quota system separates students according to their costs, while admission without affirmative action allows all the students compete in a grand contest. Besides education, the results in this paper are also applicable to a variety of competitions. For instance, in the early nineteenth century, there were no weight classes in boxing. Then eight weight classes were introduced before the Second World War, and nine more were introduced afterwards. The history of other sports such as weightlifting and wrestling also shares similar trends. The heavier athletes have an obvious advantage in strength, so why would we want to group players with similar abilities into the same class and let them compete only within their class? This takes into consideration of fairness, and our results suggest that separating athletes according to their abilities could also increase their effort and therefore make matches more entertaining.

The key characteristics common to these scenarios are: participants with potentially different abilities/costs who are divided into different groups to compete; heterogeneous prizes awarded solely on the basis of relative performance; and sunk costs of participants' investments.

This paper builds on Siegel's (2010) model of all-pay contests by introducing heterogeneous prizes and a planner who can allocate players and prize money across contests. Specifically, suppose that there is a fixed amount of prize money and a group of players with different ability types. We do not distinguish effort and performance in this paper.[5] The players of the same type have the same constant marginal cost of performance/effort. A lower cost of performance represents a higher ability. A planner can assign the players into any number of all-pay contests and divide the money as potentially heterogeneous prizes in the contests. In each contest, the players choose their performance simultaneously; the player with the highest performance receives the highest prize, the player with the second highest performance receives the second prize, and so on. The planner attaches type-specific weights to the player's performance, and wants to maximize the total weighted expected performance subject to the positive performance requirement.

Our main result is that, whatever the weights are, grouping players with similar abilities together is always superior to mixing them. Moreover, we also characterize the associated optimal prize structures. Intuitively, separating leads to the most intense competition because each player has to compete against opponents who have the same ability

[4] See Bertrand et al. (2010) for more details on quota systems in India.

[5] In other words, we assume a deterministic relation between effort and performance such that one unit of effort leads to one unit of performance.

as he does. Because of the intense competition, the players enhance their performance so much that all of them receive zero payoffs. As a result, for any set of contests with mixed players, there always exist contests of separated players with at least as much total expected performance for each ability type and a higher total expected performance for at least one ability type. In other words, having separated players dominates having mixed players in terms of the total expected performance of different types.

It is important that the planner can choose the prizes while grouping the players. In situations such as tennis or golf tournaments, choosing the prize values for different rankings is a very important part of contest design. Moreover, allowing the planner to choose prizes does not preclude the possibility that players' values of ranking are partly determined outside the tournaments. For instance, if the planner wants to have a first prize of \$200 thousand and she knows that the champion will also receive an endorsement deal worth \$100 thousand, then she can simply award the difference of \$100 thousand to the champion. A similar argument applies when a school has a budget for scholarships.

There are several challenges that we have to overcome in order to establish these results. First, with asymmetric players and heterogeneous prizes, the equilibrium of an all-pay contest may involve complicated mixed strategies. To our knowledge, equilibrium characterization for general asymmetric contests with heterogeneous prizes is still an open question. For instance, Bulow and Levin (2006) and González-Díaz and Siegel (2013) consider prizes with constant differences; Siegel (2010) studies contests with identical prizes; Xiao (2013) examines quadratic (the second-order difference in prizes is a positive constant) or geometric (the ratio of successive prizes is a constant) prize sequences; and Olszewski and Siegel (2013) consider heterogeneous prizes in large contests. In contrast, this paper allows any prize structure. The techniques developed in this paper allow us to show that contests with asymmetric players are never optimal. As a result, although we do not know the equilibrium characterization, we can still characterize the optimal way to group players.

Second, there could be multiple equilibria, and equilibrium selection may change the comparison between separating and mixing. Multiple equilibria are demonstrated in similar settings (Baye et al. 1996, Barut and Kovenock 1998). In particular, Xiao (2013) provides an example with exogenous prizes which shows that, depending on the equilibrium selection, separating may result in higher or lower total expected performance than mixing. The results in this paper apply to all equilibria, and therefore are robust to equilibrium selection.

Finally, the generality of the model also imposes extra challenges. The current paper does not restrict the number of contests, the prize structures, or the player composition in contests, which means the planner has to compare a large number of choices. Moreover, this paper accommodates a very general objective function for the planner: she can assign asymmetric type-specific weights to the players' performance, and she can impose different minimal performance requirements on different types.

Literature Baye et al. (1993) show that a politician can extract higher rent by excluding the lobbyists, who have higher valuations of winning, from the competition. This is referred to as the "exclusion principle". There are two major differences between our paper and theirs. First, they consider a model of an all-pay auction in which the lobbyists compete for a single prize. While the setup with a single prize applies to many situations besides lobbying, it is also important to accommodate the prevalence of heterogeneous prizes in sports tournaments and in competition among students. In our setup, a single prize may lead to some players' non-performance, therefore it is not optimal.[6] Our optimal prize structure requires multiple potentially heterogeneous prizes to provide incentive for all the players in each contest, so no player exhibits non-performance. Second, they consider maximization of total performance. This paper generalizes the objective so that the planner can attach different weights to different players.

Moldovanu and Sela (2006) compare different ways to group ex ante symmetric players across contests. They find that total expected effort is maximized by a grand contest including all players, while the expected highest effort is maximized by splitting players into multiple contests and letting winners of each contest compete in a final contest. Fu and Lu (2009) find that merging multi-winner contests of symmetric players can increase total expected effort. In contrast, this paper considers players with asymmetric abilities. This assumption is crucial as we are considering whether players should be separated according to their abilities.

Lazear (2001) studies tracking when students are awarded according to their absolute performance. He shows that tracking results in higher total performance than mixing. Studies on peer effects also discuss grouping players across competitions (see, for instance, Board 2009 and Cooley 2009). The players' payoffs in these papers also depend on their absolute performance, while we consider a different situation in which students are awarded according to their relative performance, or the ranking of their performance.

Studies in the setup of incomplete information are also relevant. For instance, Hickman (2009) studies different affirmative action policies on college admissions. In this paper, minority students' private costs are stochastically higher than those of majority students, and they compete for prizes/seats of exogenous values. In a quota system, the two groups compete separately for their respective reserved seats. Compared to the case without affirmative action, the quota system has unclear effects on average performance within groups and within the overall population. Parreiras and Rubinchik (2014) also consider contests with incomplete information. In their paper, each player's private ability is

[6] Remark 2 illustrates this point in an example.

uniformly distributed, and the distributions are different across players. They find that if the players are similar, the total revenue is maximized by two equal contests separating the players of higher abilities from those of lower abilities. The prizes are exogenous in both papers. In contrast, this paper considers the joint decision of grouping students and allocating prizes.

The literature on status competition studies the optimal way to divide players into different status categories when the players' payoffs depend directly on their status (see for instance Moldovanu et al. 2007 and Dubey and Geanakoplos 2010). In our paper, performance ranking does not affect players' payoffs directly. Instead, the payoffs depend on the prizes awarded according to the players' ranking.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the main results on optimal grouping and optimal prize allocations. Section 4 discusses different extensions.

2 Model

There is one unit of prize money and a set of players, N. The players are of $T \geq 2$ different types, and there are n_t players of type $t \in \{1, 2, ..., T\}$. We assume $n_t \geq 2$ for all t, so there are similar players of each type.[7] We do not distinguish effort and performance. More precisely, we assume a deterministic relation between effort and performance so that one unit of effort leads to one unit of performance. Players of t-type have the same marginal cost of performance, c_t. Without loss of generality, we assume $0 < c_1 < c_2 < ... < c_T$.

A planner's decision has two parts: assigning the players into any number of contests, and dividing the prize money as prizes for each of the contests. More precisely, the planner's choices can be represented by a partition of the players $\mathcal{P}$ and a prize structure $\mathcal{V}$. The partition $\mathcal{P}$ is a family of non-empty subsets of N such that N is a disjoint union of the subsets. Suppose partition $\mathcal{P}$ consists of m sets, then we have $\mathcal{P} = \{P_1, ..., P_m\}$. The prize structure $\mathcal{V}$ is a family of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_m\}$, where vector $\mathbf{v}_k \in [0, 1]^{|P_k|}$ and $|P_k|$ is the number of players in set P_k. A zero entry of $\mathbf{v}_k$ means one of the prizes is zero. Therefore, the partition $\mathcal{P}$ and prize structure $\mathcal{V}$ characterize m individual contests. In particular, the subset $P_k \subset N$ is the set of players assigned to contest k, and $\mathbf{v}_k$ represents the prizes in contest k. Since there is no restriction on the number of contests, the planner may assign all the players into one contest, that is, $\mathcal{P} = \{N\}$.

Let us describe the competition in each contest. In a contest characterized by P_k and $\mathbf{v}_k$, all the players in P_k choose their performance/effort levels in $[0, +\infty)$ simultaneously. The player with the highest performance receives the highest prize in $\mathbf{v}_k$; the player with

[7]This assumption is relaxed in Section 4.2.

the second highest performance receives the second highest prize in $\mathbf{v}_k$; and so on. In the case of a tie, the prizes are allocated randomly such that no tying player loses with certainty.[8] If a player wins a prize, his payoff is his prize net of his cost of performance. If a player wins no prize, his payoff is zero minus his cost of performance. All players are risk neutral. This considers Nash equilibria. A profile of strategies constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players.

The planner's objective has two parts. First, the planner attaches weight α_i to a t-type player i's expected performance, and she wants to maximize a weighted sum of expected performance. We assume that $\alpha_i \in (0,1)$ and $\Sigma_{i=1}^{T}\alpha_i = 1$. The weight α_t represents the relative importance of t-type players' performance to the planner. Second, the planner has a positive performance requirement. Let s_{i_t} be the performance of player i_t of t-type. Then, the planner wants to ensure $s_{i_t} > 0$ almost surely for all $i_t \in N$. Note that the requirement is on the planner's objective instead of players' choice. This means that the planner has to provide well-designed incentives so that the players do not choose non-performance even if they could. Because s_{i_t} is restricted to an open set, we assume the total prize in each contest has to be no less than $V_{\min} > 0$. This assumption ensures existence of an optimal prize structure.[9] In addition, we assume $V_{\min}$ is small such that $TV_{\min} \leq 1$.

Let us introduce some definitions that are important in later analysis. We say a partition and prize structure pair $(\mathcal{P}, \mathcal{V})$ is feasible if each contest has a total prize value no less than $V_{\min}$ and the total prize in $\mathcal{V}$ is no more than 1. Denote by Φ as the set of all feasible partition and prize structure pairs. Given any feasible $(\mathcal{P}, \mathcal{V})$, let $E[s_{i_t}]$ be the expected performance of player i_t of t-type in an equilibrium. Define a correspondence $\Pi : \Phi \to [0, +\infty)$ such that

$$\Pi(\mathcal{P}, \mathcal{V}) = \sum_{t=1}^{T} \left(\alpha_t \sum_{i_t=1}^{n_t} E[s_{i_t}] \right)$$

Note that $\Pi(\mathcal{P}, \mathcal{V})$ may contain multiple values if there are multiple equilibria. We say $(\mathcal{P}^*, \mathcal{V}^*)$ maximizes $\Pi(\mathcal{P}, \mathcal{V})$ if $\inf \Pi(\mathcal{P}^*, \mathcal{V}^*) \geq \sup \Pi(\mathcal{P}, \mathcal{V})$ for any feasible $(\mathcal{P}, \mathcal{V})$, where the infimum and supremum are taken over the multiple values in $\Pi(\mathcal{P}^*, \mathcal{V}^*)$ or $\Pi(\mathcal{P}, \mathcal{V})$.

The planner chooses partition $\mathcal{P}$ and prize structure $\mathcal{V}$ to maximize the weighted sum of expected performance subject to positive performance requirements. Therefore, her

[8] In many tournaments (for example, in golf), ties are resolved by sharing the prizes. As an example, if two players tie with the second-highest score, then each receives the average of the second and third prize. Our formulation allows this kind of sharing.

[9] Remark 2 discusses the consequence of removing the assumption $V_{\min} > 0$.

problem is

$$\max_{(\mathcal{P},\mathcal{V})\in\Phi} \Pi(\mathcal{P},\mathcal{V}) \tag{1}$$

$$\text{s.t.} \quad s_{i_t} > 0 \text{ a.s. for all } i_t \in N \text{ in any equilibrium} \tag{2}$$

Restriction (2) means that any player's non-performance is unacceptable to the planner. We say $(\mathcal{P}^*,\mathcal{V}^*)$ is optimal if $(\mathcal{P}^*,\mathcal{V}^*)$ maximizes $\Pi(\mathcal{P},\mathcal{V})$ and if (2) is satisfied.

3 Optimal Grouping

If a contest only has one player, his equilibrium performance level must be zero, which violates (2). Therefore, we only need to consider the partitions in which each contest contains at least two players. The planner is said to *separate* the players if each contest contains players of a same type. Otherwise, we say the planner *mixes* the players. In order to separate the players, the planner cannot have fewer than T contests, but she can have more than T by splitting a larger contests into smaller ones. The main result of this paper is as follows.

Theorem 1 *For any given weights, the total weighted expected performance is maximized only if the players are separated.*

Remark 1 *Theorem 1 is robust to small idiosyncratic shocks in the costs. More precisely, suppose player i_t of t-type has a marginal cost $c_t + \varepsilon_{i_t} > 0$, where ε_{i_t} represents the idiosyncratic shock and is commonly known. Then, there exists an $\varepsilon > 0$ such that Theorem 1 remains true if $\max_{i_t\in N} |\varepsilon_{i_t}| < \varepsilon$. See Proposition 5 in Appendix.*

Remark 2 *If the planner does not have the restriction of $s_{i_t} > 0$ almost surely, then separating becomes weakly better than mixing. More precisely, the maximum total weighted expected performance can be achieved if the players are separated, and it can also be achieved if players are mixed. See Example 1. We can also obtain Theorem 1 if we replace the positive performance requirement by the assumption that each contest contains distinct prizes. This assumption means that the planner has to award higher performance with a higher prize.*

In the remainder of this section, we first prove Theorem 1 through a sequence of lemmas, then specify the associated optimal prize structure in Proposition 1. The first lemma ensures equilibrium existence.

Lemma 1 *In each contest, there exists no Nash equilibrium in pure strategies, but there exists a Nash equilibrium in mixed strategies.*

Siegel (2010) establishes equilibrium existence for contests with homogeneous prizes, and his proof is readily adapted to prove the above lemma.

One of the challenges is that there may be multiple equilibria in a contest. Our method relies only on the properties that are true for any equilibrium, thus we overcome this challenge. The lemmas below present four such properties of any equilibrium. Lemma 2 characterizes an important property of atoms in one's equilibrium strategy, and we use this lemma to prove Lemma 3.

Lemma 2 *Suppose a player has an atom at performance level s in an equilibrium, that is, he chooses s with a strictly positive probability. Then he loses with certainty by choosing this performance level.*

The proof is in Appendix. Consider a contest with a player set P_k and a prize vector $\mathbf{v}_k$. We use a cumulative distribution function $G_i : [0, +\infty) \to [0, 1]$ to represent player i's mixed strategy. Player i chooses pure strategy s_i if G_i assigns probability 1 to s_i. In the contest, given other players' strategies $\mathbf{G}_{-i} \equiv (G_j)_{j \in P_k \setminus \{i\}}$, we denote player i's expected winnings by choosing s as $W(\mathbf{G}_{-i}(s), \mathbf{v}_k)$.

Lemma 3 *Consider a contest with a player set P_k and a prize vector $\mathbf{v}_k$, in which not all the prizes are identical. Given any equilibrium, let $\bar{s}_j$ be player j's highest performance in the support of his equilibrium strategy. Then,*

$$W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_k) \geq W(\mathbf{G}^*_{-j}(\bar{s}_j), \mathbf{v}_k) \tag{3}$$

Proof. There are two possibilities. First, suppose $\bar{s}_j = 0$. It means player j chooses performance level 0 with probability 1, then Lemma 2 implies that player j loses with certainty. Hence, $W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_k)$ cannot be lower than j's expected winnings in the equilibrium. That is, inequality (3) holds.

Second, suppose $\bar{s}_j > 0$. Lemma 2 implies that no player chooses $\bar{s}_j$ with positive probability in the equilibrium. Therefore, if player i chooses performance level $\bar{s}_j$, his performance is higher than j's with certainty. In other words, by choosing $\bar{s}_j$, player i beats j for sure. In addition, they face the same competition from the other players. Hence, if both i and j choose $\bar{s}_j$, player i's expected winnings is no less than that of j's. That is, inequality (3) holds. ■

A mixed strategy's support is the smallest closed set that receives probability 1 according to the strategy.

Lemma 4 *If all the players in a contest are identical, each player's expected payoff in any equilibrium must equal to the value of the lowest prize.*[10]

Proof. Given any equilibrium, let $\underline{s}$ be the lowest performance level in the supports of the mixed strategies. Then, at least one player's mixed strategy has $\underline{s}$ as the lower bound of its support. If $\underline{s} > 0$, this player wins the lowest prize with performance $\underline{s}$. On the other hand, he could also win the same prize with performance 0, which incurs a lower cost. This is a contradiction. As a result, we must have $\underline{s} = 0$, and the payoff of this player equals the lowest prize. We will show below that no player can have a different payoff.

If the prizes are identical, everyone receives the same prize hence the lemma is true. Suppose the prizes are not identical, and suppose player i's payoff equals the lowest prize and player j's payoff in an equilibrium is higher than the lowest prize. Let $\bar{s}_j$ be the highest performance level in the support of player j's mixed strategy and $\bar{s}_i$ be the counterpart for i. Lemma 3 implies that player i's expected winnings with performance level $\bar{s}_j$ is no lower than player j's. In addition, notice that both players have the same marginal cost, so player i's payoff at $\bar{s}_j$ is no lower than that of j at $\bar{s}_j$. This contradicts the assumption that player j's payoff is higher than i's. ■

Lemma 5 *If the players in a contest are not identical and the prizes are different, at least one player's payoff is higher than the lowest prize in any equilibrium.*

Proof. Suppose players i and j have different costs, with $c_i > c_j$. Since the payoffs cannot be lower than the lowest prize, it is sufficient to show that players i and j have different payoffs. Assume otherwise that $u_j = u_i$, where u_i and u_j are i and j's expected payoffs in an equilibrium. According to Lemma 3, player j's expected winnings at $\bar{s}_i$ is no lower than i's at $\bar{s}_i$. Therefore, if player j deviates to $\bar{s}_i$, his expected winnings is not lower than i's, but his cost is lower than i's. Hence, the deviation results in j's payoff higher than u_i. This is a contradiction because i and j have the same payoff by assumption. ■

Now we can proceed to prove Theorem 1. Given any equilibrium, denote by $S_t \equiv \sum_{i=1}^{n_t} E[s_{i_t}]$ as the total expected equilibrium performance of all t-type players. Intuitively, we first show that any performance outcome $(S_1, ..., S_T)$ with mixed players is *dominated* by an outcome $(S_1', ..., S_T')$ with separated players in the sense that $S_i \leq S_i'$ for all i and $S_i < S_i'$ for some i. Then, we show that with the partition and prize structure associated with $(S_1', ..., S_T')$, there are no other outcomes. This shows that separating is superior to mixing for any type-specific weights.

[10] Corollary 3 of Siegel (2009) proves this lemma for the case of identical prizes. However, the argument for identical prizes does not apply here directly. For instance, the definition of "power", an important concept in Siegel's analysis, does not apply to the case of heterogeneous prizes.

Proof of Theorem 1. Suppose the players are mixed. Consider an equilibrium in which the positive performance requirement is satisfied. Then, we must have $E[s_{i_t}] > 0$ for any i_t. Given the equilibrium, let $(S_1, ..., S_T)$ be the performance outcome, U_t be t-type players' total expected payoff, and W_t be their total expected winnings. For each player, his payoff equals his expected winnings net of the cost of his expected performance, so we have $U_t = W_t - c_t S_t$, which gives us the expression of his expected performance

$$S_t = (W_t - U_t)/c_t \tag{4}$$

Note that U_t is the total payoff when the players are mixed. Lemma 5 implies that $U_t > 0$ for some t, so we also have

$$\Sigma_t(W_t - U_t) < \Sigma_t W_t = 1 \tag{5}$$

Suppose that the planner separates the players into T contests, so each contest contains all the players of a particular type. In addition, suppose she assigns prizes of total value $W_t - U_t$ to the contests with t-type players and all the prizes except the lowest are positive. Then, the total winnings of t-type players equals the total value of their prizes, $W_t - U_t$. If the lowest prize is zero in every contest, Lemma 4 implies that all players have zero payoffs. Similar to (4), the total expected performance of t-type players is $((W_t - U_t) - 0)/c_t = S_t$, and the performance outcome is the same as $(S_1, S_2, ..., S_T)$. According to (5), some prize money, $1 - \Sigma_t(W_t - U_t)$, is not assigned to any contest. If the planner adds the extra money to the first prize in a contest of t-type players, the total performance of t-type would increase. Hence, the resulting performance outcome dominates $(S_1, S_2, ..., S_T)$.

According to Barut and Kovenock (1998), if all the players in a contest are identical and all the prizes except the lowest are positive, the contest has a unique equilibrium. Therefore, if players are separated, the outcome is unique. Hence, for any prize structure and any partition that mixes players, we find another partition and prize structure such that the outcome is unique and the outcome dominates that associated with the mixing partition. As a result, the total weighted performance is never maximized if the players are mixed.

To complete the proof, we still need to ensure that $s_{i_t} > 0$ almost surely for any player i_t. According to Barut and Kovenock (1998), if the players are identical and if all the prizes except the lowest are positive, the unique equilibrium is symmetric in the contest. Therefore, (2) is satisfied. ■

According to the theorem, it is never optimal to assign only one player to a contest or to have a contest containing all the players. It is also worth mentioning that there is more than one way to separate the players. For instance, suppose there are two H-type players and four L-type players. The planner can separate the players in two ways. She can have two contests with all the H-type players in one and all the L-type players in the other. Alternatively, she can have one contest with all the H-type players and two other identical contests, each with two L-type players. Proposition 1 below implies that the optimal performance outcome remains the same across the different ways of separating.

Now let us consider the optimal prize structure. As a result of Theorem 1, we only need to find the optimal prizes for separated players. According to Lemma 4, if the lowest prize in a contest becomes smaller, the total expected payoff in the contest decreases, therefore the total expected performance increases because of (4). Hence, the lowest prize should be zero in every contest, then all players should have zero payoffs. Therefore, equation (4) implies that the total performance of t-type players is V_t/c_t, where V_t is the total value of prizes assigned to the contests containing t-type players. Note that, if the players are separated, the distribution of prize money within a contest has no effect on the total performance in the contest as long as all the prizes except the lowest remain positive.[11] As a result, the planner's problem (1) becomes a linear programming problem

$$\max_{V_1,..,V_T \geq V_{\min}} \quad \Sigma_t(\alpha_t V_t/c_t) \qquad \text{s.t.} \quad \Sigma_t V_t = 1 \tag{6}$$

If $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$, it is optimal to minimize V_t, so $V_t = V_{\min}$. Based on the analysis above, the proposition below characterizes the optimal prize structures that solve the planner's problem.

Proposition 1 *A prize structure is optimal for separated players if and only if i) all the prizes except the lowest are positive in each contest, ii) the total value of prizes in all the contests of t-type players is $V_t = V_{\min}$ if $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$.*

Remark 3 *A single prize, as in an all-pay auction, is not optimal if a contest contains more than two players. For instance, if there are three H-type players and two L-type players, the only way to separate them is grouping all the H-type players in one contest and both L-type players in another. Then, if we assign a single prize in the contest with three H-type players, there exists an equilibrium in which one player chooses zero performance level. Therefore, the positive performance requirement is violated.*

There could be more than one optimal prize structure, but they all result in the same performance outcome. Moreover, because Lemma 4 applies to any equilibrium, the optimal prize structures and the induced performance outcome remain the same whether there are multiple equilibria or not.

[11] In different setups where the participants' costs are private information, allocation of prizes would affect the equilibrium performance. See, for example, Moldovanu et al. (2007) and Liu et al. (2013).

So far we have demonstrated that for separating to be optimal, it should be accompanied with associated optimal prize structures. The share of the budget that is used to motivate the players of t-type is weakly increasing in the weight α_t. If the planner cannot choose the prizes freely, mixing could actually be better than separating. For example, if the school cannot ensure enough scholarships or resources are allocated to the lower-ability groups, the students in those groups could have academic achievements below the minimal requirements. Then, it could be beneficial to mix the students with different abilities.

If the planner does not have the restriction of $s_{i_t} > 0$ almost surely, separating becomes weakly better than mixing. Let us illustrate it in an example.

Example 1 *Suppose there are two players with c_1 and four with c_2. We also suppose $\alpha_1 = \alpha_2$, so the planner wants to maximize the total performance.*

If we put all the players with c_1 in one classroom and the others in another classroom, and if we allocate all the budget to a single prize in the classroom with marginal cost c_1, the total weighted performance is maximized. Moreover, both players with cost c_2 choose non-performance. However, the total performance can also be maximized by mixing the players. If we put all the players in one classroom and allocate the budget to a single prize. The two players with c_1 compete for the prize, and the other two choose non-performance. Therefore, even if the players are assign to the same classroom, they compete as if they are separated. As a result, the total expected performance is also maximized in a mixed classroom.

4 Extensions

4.1 Minimal Effort Requirements

In the previous section, if $V_{\min}$ – the minimum total prize in a contest – is close to zero, a player's expected performance could also be close to zero. This section considers an additional objective of the planner to avoid the small expected performance. In particular, we assume $V_{\min} = 0$ and replace the positive performance requirement $s_{i_t} > 0$ a.s. by $E[s_{i_t}] \geq r_t$ for some $r_t > 0$.[12] That is, a t-type player's expected performance should be at least $r_t > 0$ in any equilibrium. Note that $r_1, ..., r_T$ need not be the same. As a result, the restriction in (1) becomes $E[s_{i_t}] \geq r_t$ for $i_t = 1, ..., n_t$ and $t = 1, ..., T$. If $\Sigma_t r_t / c_t > 1$, the minimal levels for the expected performance cannot be achieved. This is because the total cost of achieving the minimal levels is higher than the total value of the prizes. Therefore, we assume $\Sigma_t r_t / c_t \leq 1$. The following result extends Theorem 1 to accommodate the minimal requirements of expected performance.

[12] The analysis is the same for a $V_{\min}$ close to zero.

Proposition 2 *For any given weights and minimal expected performance levels, the total weighted expected performance is maximized only if the players are separated.*

The proof is similar to that of Theorem 1, so it is omitted. Now we consider the optimal prize structure. Problem in (6) becomes

$$\begin{aligned} \max_{V_1,..,V_T \geq 0} \quad & \Sigma_t \alpha_t V_t / c_t \\ \text{s.t.} \quad & \Sigma_t V_t = 1 \\ & V_t/(c_t n_t) \geq r_t \text{ for all } t \end{aligned}$$

If $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$, it is optimal to minimize V_t, so $V_t = c_t n_t r_t$, which is just enough to maintain the minimal performance requirement. In addition, we also need to ensure the minimal effort requirements are met. Recall that each player in a contest of identical players exhibits the same expected performance. Therefore, in order to ensure the minimal performance requirements are met, the per capita prize should be at least $c_t r_t$ in each contest containing t-type players. Based on the analysis above, the proposition below characterizes the optimal prize structures that solve the planner's problem.

Proposition 3 *A prize structure is optimal for separated players if and only if i) all the prizes except the lowest are positive in each contest, ii) the total value of prizes in all the contests of t-type players is $V_t = c_t n_t r_t$ if $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$, and iii) the total value of prizes in a contest containing k_t t-type players is at least $c_t k_t r_t$.*

Remark 4 *It is worth of mentioning that, given the optimal partition and prize structure, the previous restriction (2) is also satisfied.*

4.2 Distinct Abilities

This section relaxes the assumption $n_t \geq 2$ and considers players with distinct marginal costs. In particular, suppose that there are four players with marginal costs $0 < c_1 < c_2 < c_3 < c_4$. Moreover, $\alpha_t = 1/4$ for $t = 1, 2, 3$ and 4, which means the planner wants to maximize the total expected performance. Because it is never optimal to have one player in a contest, there are three ways to group the players: $(\{1, 2\}, \{3, 4\}), (\{1, 3\}, \{2, 4\})$ and $(\{1, 4\}, \{2, 3\})$. Let v_{ij} be the prize in the contest of players $\{i, j\}$. We relax the assumption of positive performance requirement and remove the minimal total prize in each contest, but the analysis is similar with these assumptions. The following proposition characterizes the optimal partition and prize structure.

Proposition 4 *Consider four players with marginal costs of $0 < c_1 < c_2 < c_3 < c_4$. If $(c_2 + c_3)/c_3^2 \geq (c_1 + c_2)/c_2^2$ and $(c_2 + c_3)/c_3^2 \geq (c_3 + c_4)/c_4^2$, the optimal partition is*

$(\{1,4\},\{2,3\})$ *and optimal prize structure is* $v_{14}=0$ *and* $v_{23}=1$. *Otherwise, the optimal partition is* $(\{1,2\},\{3,4\})$, *and the optimal prize is* $v_{12}=1$ *if* $(c_1+c_2)/c_2^2 > (c_1+c_2)/c_2^2$ *and* $v_{34}=1$ *if* $(c_1+c_2)/c_2^2 < (c_3+c_4)/c_4^2$.

The proof is in Appendix. The proposition implies that players with similar abilities should be grouped together. It is worth mentioning that grouping players with similar abilities together does not necessarily mean grouping players with higher abilities together. On one hand, they could represent the same allocation of players. For instance, given c_1 and c_4, if c_3-c_2 converges to c_4-c_1, player 2's marginal cost converges to 1's, and player 4's converges to 3's. Then, Proposition 4 implies that the similar players should be grouped together, which in this particular case also means the players with higher abilities are grouped together. On the other hand, if c_2 and c_3 are similar and if c_2-c_1 and c_4-c_3 are large, Proposition 4 implies that it is optimal to assign 2 and 3 into one contest and 1 and 4 into another. Therefore, grouping players with higher ability together is not optimal.

It would be an interesting extension to consider the optimal grouping problem for more than four players whose marginal costs are all different. As in Proposition 4, the optimal grouping would depend on the distribution of costs. Moreover, since each contest inevitably has asymmetric players, it is very important to characterize the equilibria in asymmetric contests with heterogeneous prizes, which to our knowledge is still an open question. Similarly, extending the model to accommodate constraints on the maximum number of contests or on the number of participants in each contest would lead to the same challenge in equilibrium characterization.

4.3 Heterogeneous Valuations

Our analysis can be extended to allow heterogeneous valuation of the same prize. Consider a contest with players $1,...,n$ and prizes $v_1 \geq ... \geq v_n \geq 0$. Player i is characterized by $(d_i,x_i)\in(0,\infty)^2$. His value of the kth prize is $d_i v_k$ for $k=1,...,n$, and his marginal performance cost is x_i. Parameters $d_1,...,d_n$ represent players heterogeneous values. Consider another contest with n players and the same prizes. In this contest, player i has marginal cost $\hat{c}_i \equiv x_i/d_i$ and his value of the kth prize is v_k. Then, the contest is the same as in Section 2. Given the same prize, the payoff of player with (d_i,x_i) in the first contest is the payoff of player $\hat{c}_i$ in the second contest multiplied by d_i. Therefore, the two contests have the same set of equilibria. Hence, we can replace c_i in Section 2 by x_i/d_i, and the analysis remains the same.

4.4 Other Planner's Objectives

Our analysis applies to other objectives of the planner. Suppose $T = 2$, so there are two ability types. Suppose the planner's objective is

$$\max_{(\mathcal{P},\mathcal{V})\in\Phi} (S_1 + S_2) - \beta |S_1/n_1 - S_2/n_2| \tag{7}$$

subject to the restriction (2), where $\beta \in (0, 1)$. This means that the planner wants to maximize the total expected performance and she also wants to minimize the difference in the average performance across groups. Because $\beta < 1$, maximizing total performance is more important than minimizing the performance gap. A similar objective is considered by Chan and Eyster (2003) in a study of college admission policies. We can verify that (7) s.t. (2) has the same optimal partition and prize structure either as $\max(1 - \beta/n_1)S_1 + (1+\beta/n_2)S_2$ subject to (2) or as $\max(1+\beta/n_1)S_1+(1-\beta/n_2)S_2$ subject to (2). In either case, we transform (7) into a problem as in (1), so the analysis in Section 2 applies.

References

Barut, Y. and Kovenock, D. (1998), "The Symmetric Multiple Prize All-Pay Auction with Complete Information", *European Journal of Political Economy*, 14, pp. 627-644. (Not cited.)

Baye, M. R.; Kovenock, D. and de Vries, C. G. (1993), "Rigging the Lobbying Process: An Application of the All-Pay Auction", *American Economic Review*, 83, pp. 289-294. (Not cited.)

Baye, M. R.; Kovenock, D. and de Vries, C. G. (1996), "The All-Pay Auction with Complete Information", *Economic Theory*, 8, pp. 291-305. (Not cited.)

Bertrand, M.; Hanna, R. and Mullainathan, S. (2010), "Affirmative Action in Education Evidence from Engineering College Admissions in India", *Journal of Public Economics*, 94, pp. 16-29. (Not cited.)

Board, S. (2009), "Monopolistic Group Design with Peer Effects", *Theoretical Economics*, 4, pp. 89-125. (Not cited.)

Bulow, J. and Levin, J. (2006), "Matching and Price Competition", *American Economic Review*, 96, pp. 652-668. (Not cited.)

Chan, J. and Eyster, E. (2006), "Does Banning Affirmative Action Lower College Student Quality?" *American Economic Review*, 93, pp. 858-872. (Not cited.)

Cooley, J. (2010), "Desegregation and the Achievement Gap: Do Diverse Peers Help?" Working paper. (Not cited.)

Dubey, P. K. and Geanakoplos, J. (2010), "Grading Exams: 100, 99, 98, ... or A, B, C?" *Games and Economic Behavior*, 69, pp. 72-94. (Not cited.)

Duflo, E.; Dupas, P. and Kremer, M. (2011), "Peer Effects and the Impacts of Tracking: Evidence from a Randomized Evaluation in Kenya", *American Economic Review*, 101, pp. 1739-1774. (Not cited.)

Fu, Q. and Lu, J. (2009), "The Beauty of 'Bigness': On Optimal Design of Multi-Winner Contests", *Games and Economic Behavior*, 66, pp. 146-161. (Not cited.)

González-Díaz, J. and Siegel, R. (2013), "Matching and Price Competition: Beyond Symmetric Linear Costs", *International Journal of Game Theory*, 42, pp. 835-844. (Not cited.)

Hickman, B. R. (2009), "Effort, Race Gaps, and Affirmative Action: A Game-Theoretic Analysis of College Admissions", Working paper. (Not cited.)

Lazear, E. P. (2001), "Educational Production", *Quarterly Journal of Economics*, 116, pp. 777-803. (Not cited.)

Liu, B.; Lu, J.; Wang, R.; and Zhang, J. (2013), "Prize and Punishment: Optimal Contest Design with Incomplete Information", Working paper. (Not cited.)

Loveless, T. (2013), "The Resurgence of Ability Grouping and Persistence of Tracking", *Part II of the 2013 Brown Center Report on American Education*, the Brookings Institution. (Not cited.)

Moldovanu, B. and Sela, A. (2006), "Contest Architecture", *Journal of Economic Theory*, 126, pp. 70-96. (Not cited.)

Moldovanu, B.; Sela, A. and Shi, X. (2007), "Contests for Status", *Journal of Political Economy*, 115, pp. 338-363. (Not cited.)

Olszewski, W. and Siegel, R. (2013), "Large Contests", Mimeo, Northwestern University. (Not cited.)

Parreiras, S. O. and Rubinchik, A. (2014), "Group Composition in Contests", Mimeo, University of North Carolina. (Not cited.)

Siegel, R. (2009), "All-Pay Contests", *Econometrica,* 77, pp. 71-92. (Not cited.)

Siegel, R. (2010), "Asymmetric Contests with Conditional Investments", *American Economic Review*, 100, pp. 2230–60. (Not cited.)

Xiao, J. (2013), "Asymmetric All-Pay Contests with Heterogeneous Prizes", Mimeo, University of Melbourne. (Not cited.)

Yee, V. (2013), "Grouping Students by Ability Regains Favor in Classroom", *New York Times*, June 9, 2013. (Not cited.)

Appendix

Proof of Lemma 2. We first claim that, if two or more players have an atom at performance level s in an equilibrium, all the players who have an atom at s lose with certainty.[13] Let us prove it by contradiction. Suppose that two players, i and j, have an atom at performance level s in an equilibrium, and suppose that player i wins a prize with positive probability by choosing s. Since the tie is broken in such a way that everyone involved wins with positive probability, player j also wins a prize with positive probability by choosing s. In addition, the tie breaking rule ensures that player i loses with positive probability by choosing s, so he does not win the highest prize with probability 1. In contrast, if player j increases his performance slightly above s, his cost is almost the same but his expected winnings would have a discontinuous increase. This is because he no longer needs to share any prize with player i. This is a deviation for player j, which is a contradiction.

We prove Lemma 2 in two steps. First, suppose two players have an atom at performance level s in the equilibrium, then the above claim implies that both of them must lose with certainty by choosing s. Second, suppose only player i has an atom at s, and suppose he wins a prize with positive probability. On the one hand, if all other players have no best response in $(s-\varepsilon, s)$ for some $\varepsilon > 0$, player i would benefit from lowering the atom to $s-\varepsilon$. This is a contradiction. On the other hand, suppose another player j has a sequence of best responses converging to s from below. Compared to such a best response close to s, performance slightly above s imposes an almost identical cost on player j, but the resulting expected winnings would have a discontinuous increase because of player i's atom at s. This is also a contradiction. In sum, player i loses with certainty by choosing performance level s, which completes the proof. ■

Proposition 5 *Suppose player i_t of t-type has a marginal cost $c_t + \varepsilon_{i_t} > 0$, where ε_{i_t} represents the idiosyncratic shock and is commonly known. Then, for any weights and minimal performance requirements, there exists an $\varepsilon > 0$ such that if $\max_{i_t \in N} |\varepsilon_{i_t}| < \varepsilon$, the total weighted expected performance is maximized only if the players are separated.*

[13] This claim is referred to as the Tie Lemma by Siegel (2009).

Proof. Let us first show the counterpart of Lemma 4: the players' payoffs in a contest converge to the value of the lowest prize in any equilibrium if $\max_{i_t \in N} |\varepsilon_{i_t}|$ goes to zero. Similar to Lemma 4, the player with the highest marginal cost, say player n, has a payoff that equals to the lowest prize. According to Lemma 3, player n can ensure himself expected winnings no less than that of player i's at performance level $\bar{s}_i$, player i's highest performance in the support of his strategy. As a result, if players n and i's costs converge towards each other, player n's payoff cannot be lower than i's in the limit. Since no player's equilibrium can be lower than the lowest prize, the payoffs of player i and n must be the same and equal to the lowest prize in the limit.

Let us now show the counterpart of Lemma 5: if the players in a contest have different cost types and the prizes are not identical, at least one player's payoff is higher than the lowest prize in any equilibrium. Suppose players i and j have different cost types, then their costs in the limit are also different: $c_i > c_j$, also suppose that they have the same equilibrium payoff in the limit, that is, $u_i = u_j$. According to Lemma 3, player j can ensure himself expected winnings no lower than that of player i's at performance level $\bar{s}_i$. Therefore, player j can ensure himself a payoff higher than i's by choosing $\bar{s}_i$ because j has a lower cost in the limit. This is a contradiction.

Given the counterparts of Lemmas 4 and 5, we can prove Proposition 2 in the exact same way as we prove Theorem 1. ■

Proof of Proposition 4. Suppose the partition is $(\{1,2\},\{3,4\})$, which means players 1 and 2 are in one contest, and players 3 and 4 in another. Then, player 1 and 2 compete in a contest for a prize of $v_{12} \in (0,1)$, and players 3 and 4 compete in the other contest for a prize of $1 - v_{12}$. The equilibrium payoffs are $u_1 = v_{12}(1 - c_1/c_2)$ for player 1 and $u_2 = 0$ for player 2. The equilibrium strategies are $G_1(s) = (u_1 + c_1 s)/v_{12}$ and $G_2(s) = c_2 s / v_{12}$. Hence, the expected performance of the two players are

$$
\begin{aligned}
E[s_1] &= \int_0^{v_{12}/c_2} s dG_1(s) = \frac{v_{12}}{2c_2} \\
E[s_2] &= \int_0^{v_{12}/c_2} s dG_2(s) = \frac{v_{12} c_1}{2c_2^2}
\end{aligned}
$$

Similarly, the expected performance for players 3 and 4 are $E[s_3] = (1 - v_{12})/(2c_4)$ and $E[s_4] = (1 - v)c_3/(2c_4^2)$. Therefore, the maximum total expected performance given the partition is

$$
\Pi_{12,34} = \max_{v_{12} \in [0,1]} \left(\frac{v_{12}}{2} \frac{c_1 + c_2}{c_2^2} + \frac{1 - v_{12}}{2} \frac{c_3 + c_4}{c_4^2} \right) = \frac{1}{2} \max \left(\frac{c_1 + c_2}{c_2^2}, \frac{c_3 + c_4}{c_4^2} \right)
$$

Similarly, the maximum expected performance with other partitions are

$$\Pi_{13,24} = \max_{v_{13}\in[0,1]} \left(\frac{v_{13}}{2} \frac{c_1 + c_3}{c_3^2} + \frac{1 - v_{13}}{2} \frac{c_2 + c_4}{c_4^2} \right) = \frac{1}{2} \max \left(\frac{c_1 + c_3}{c_3^2}, \frac{c_2 + c_4}{c_4^2} \right)$$

$$\Pi_{14,23} = \max_{v_{14}\in[0,1]} \left(\frac{v_{14}}{2} \frac{c_1 + c_4}{c_4^2} + \frac{1 - v_{14}}{2} \frac{c_2 + c_3}{c_3^2} \right) = \frac{1}{2} \max \left(\frac{c_1 + c_4}{c_4^2}, \frac{c_2 + c_3}{c_3^2} \right)$$

Hence, the maximum expected performance across all partitions is

$$\begin{aligned} & \max\left(\Pi_{12,34}, \Pi_{13,24}, \Pi_{14,23}\right) \\ = & \frac{1}{2} \max \left(\frac{c_1 + c_2}{c_2^2}, \frac{c_3 + c_4}{c_4^2}, \frac{c_1 + c_3}{c_3^2}, \frac{c_2 + c_4}{c_4^2}, \frac{c_1 + c_4}{c_4^2}, \frac{c_2 + c_3}{c_3^2} \right) \\ = & \frac{1}{2} \max \left(\frac{c_1 + c_2}{c_2^2}, \frac{c_3 + c_4}{c_4^2}, \frac{c_2 + c_3}{c_3^2} \right) \end{aligned}$$

where the second equality comes from $\frac{c_1+c_3}{c_3^2} < \frac{c_2+c_3}{c_3^2}$ and $\frac{c_1+c_4}{c_4^2} < \frac{c_2+c_4}{c_4^2} < \frac{c_3+c_4}{c_4^2}$. Therefore, partition $(\{1,3\},\{2,4\})$ is never optimal. Moreover, if $\frac{c_2+c_3}{c_3^2} > \frac{c_1+c_2}{c_2^2}$ and $\frac{c_2+c_3}{c_3^2} > \frac{c_3+c_4}{c_4^2}$, $\Pi_{14,23} > \Pi_{12,34}$. Otherwise, $\Pi_{14,23} \leq \Pi_{12,34}$. ■