All-Pay Contests with Performance Spillovers*

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Abstract

This paper generalizes the results of Siegel (2009, 2010) to accommodate performance spillovers, with which a player's performance in a contest may affect the performance cost of another player. More precisely, we show that, if for any player, spillover from other players' performance enters his cost in an additively separable form, then an all-pay contest with spillovers has a unique Nash equilibrium. Moreover, we construct the equilibrium payoffs and strategies. Both the equilibrium uniqueness and construction are generalized to multiplicatively separable spillover in a two-player contest.

JEL classification: D72, D44, L22

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1 Introduction

Performance spillovers are prevalent in contest situations. For example, higher expenditure from a lobbyist may make it easier for another lobbyist to justify his expenditure; a company's R&D effort may benefit its rivals, and hard working classmates make it easier, or less costly, for an individual student to study hard. Siegel (2009, 2010) studies contests among asymmetric players without spillovers, and Baye et al. (2012) study contests between two symmetric players with spillovers. The two setups demonstrate different equilibrium properties. For example, an asymmetric contest without spillovers has a unique Nash equilibrium, while a symmetric contest with spillovers may have one or more Nash equilibria depending on parameter values. To bridge the gap between these studies, this paper investigates contests that allow spillovers among asymmetric players. More specifically, we introduce a general class of contests with performance spillovers, which have a unique Nash equilibrium. If we introduce performance spillovers into a contest, the original equilibrium strategies may no longer be an equilibrium.[1] However, we manage to construct the equilibrium payoffs and strategies for this class of contests.

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[1] See Remark 1.

Both equilibrium uniqueness and characterization are useful for applications involving contest design in the presence of spillovers.

2 Additively Separable Spillovers

Our model builds on that of Siegel (2010), to which we add the possibility of performance spillovers. Consider a contest in which n risk neutral players compete for m homogeneous monetary prizes, where $0 < m < n$.[2] The prize value is normalized to 1.[3] Denote the set of players as $N = \{1, ..., n\}$. Each player i simultaneously chooses a performance level, or score, $s_i \geq 0$. Let $\mathbf{s} = (s_i)_{i \in N}$ be the scores of all players, and $\mathbf{s}_{-i} = (s_j)_{j \in N \setminus \{i\}}$ be the scores of all players except i. Given all players' scores $\mathbf{s}$, player i's payoff is $u_i(\mathbf{s}) = P_i(\mathbf{s}) - C_i(\mathbf{s})$, where $P_i : \mathbb{R}^n_+ \to [0, 1]$ is player i's probability of winning, and $C_i : \mathbb{R}^n_+ \to \mathbb{R}$ is his cost of score. Note whether he wins or not, player i incurs the cost.[4] The probability of winning is $P_i(\mathbf{s}) = 1$ if i's score s_i is higher than those of at least $n - m$ other players, $P_i(\mathbf{s}) = 0$ if s_i is lower than those of at least m others, and $P_i(\mathbf{s})$ equals any value in $[0, 1]$ otherwise. For each i, $C_i(\mathbf{s})$ is strictly increasing in s_i, meaning player i's score s_i is costly for him. Note that $C_i(\mathbf{s})$ depends on all players' scores, so there may be spillovers. If $C_i(\mathbf{s})$ is independent of $\mathbf{s}_{-i}$, there are no spillovers, and our setup reduces to that of Siegel (2010).

We assume that the spillover from other players' score enters the cost in an *additively separable* way, i.e., $C_i(\mathbf{s}) = K_i(s_i) + H_i(\mathbf{s}_{-i})$ for each i, where $K_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $H_i : \mathbb{R}^{n-1}_+ \to \mathbb{R}$ may differ among players, representing asymmetry in costs and spillovers respectively. The contest is of complete information, so these functions are commonly known. Recall that $C_i(\mathbf{s})$ is strictly increasing in s_i and $H_i(\mathbf{s}_{-i})$ is independent of s_i, so $K_i(s_i)$ is also strictly increasing in s_i. Then, define player i's reach as $r_i = K_i^{-1}(1)$, and re-index the players such that $r_1 \geq ... \geq r_n$.[5] We assume $r_i \neq r_{m+1}$ for $i \neq m + 1$. In addition, assume $K_i(0) = 0$ and K_i is continuous and piecewise analytic on $[0, r_{m+1}]$.[6] Moreover, for each $j \neq i$, $H_i(\mathbf{s}_{-i})$ is piecewise continuous in s_j on $[0, r_{m+1}]$.[7] The above contest is referred to as the contest with additive spillover. The following example illustrates the general model in a linear setup.

[2]Our results can be extended to heterogeneous prizes. For example, Bulow and Levin (2006) and González-Díaz and Siegel (2013) study contests with arithmetic prize sequences (with constant first order differences), and Xiao (2016) studies contests with quadratic prize sequences (with constant second order differences) or geometric prize sequences (with constant ratios between two consecutive prizes). Equilibrium uniqueness and construction are established in those contests. By the same argument in this paper, we can generalize those results to the case of additively separable spillovers.

[3]Our analysis can be extended to allow players to have asymmetric valuations of the prize.

[4]Because of the all-pay feature, the cost is sunk, so it remains the same whether a player wins. As a result, the spillovers represented by the cost functions also remain the same whether a player wins or not. In contrast, Baye et al. (2012) also consider rank-order spillovers that depend on the rank of a player's score, and demonstrate possibly multiple equilibria in the presence of rank-order spillovers.

[5]The definition of "'reach" is first introduced by Siegel (2009).

[6]A function is piecewise analytic on an interval if the interval can be partitioned into a finite number of closed intervals such that the restriction of the function to each interval is analytic.

[7]A function is piecewise continuous on an interval if the function is continuous on all points in the interval except a finite number of points at which the function has finite limits.

Example 1 *Suppose the cost is $C_i(\mathbf{s}) = c_i s_i - h\bar{s}$, where $c_i \in \mathbb{R}_+$ is player i's marginal cost of score, and $\bar{s} = (\sum_{i=1}^{n} s_i)/n$ is the average score. Behavior paper? Here the spillover depends on the average score, and h measures the scale of spillover.*[8] *If $h = 0$, there is no spillover. If h is positive (negative), a higher average score makes player i's score less (more) costly. Assume distinct marginal costs so that $0 < c_1 < ... < c_n$.*[9] *In this example, $K_i(s_i) = (c_i - h/n)s_i$ and $H_i(\mathbf{s}_{-i}) = -h(\sum_{j \neq i} s_j)/n$.*[10] *Both functions depend on the spillover parameter h. If h is positive (negative), $K_i(s_i)$ is lower (higher) than player i's scoring cost $c_i s_i$.*

A strategy profile constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players. We only consider Nash equilibria here.

Equilibrium Characterization In the absence of spillovers, the method of Siegel (2009) can be used to derive equilibrium payoffs, with which equilibrium strategies can be constructed according to the algorithm of Siegel (2010). However, this approach is not applicable here. This is because with spillovers, we can no longer derive equilibrium payoffs as in the case without spillovers.

In contrast to Siegel's method, our method first constructs equilibrium strategies, which we then use to derive equilibrium payoffs. Given the contest introduced in Section 3, consider an auxiliary contest with the same prizes but different players, whose cost functions are $K_i(s_i)$ for all i. The auxiliary contest has no spillover, but it is different from the original contest without spillovers. For instance, if $h = 0$ in Example 1, there is no spillover, and a player's scoring cost is $c_i s_i$, which is different from $K_i(s_i) = (c_i - h/n)s_i$ in the auxiliary contest.

According to Siegel (2010), the auxiliary contest has a unique equilibrium. In this contest, let $G_i : \mathbb{R}_+ \to [0, 1]$ be the c.d.f. representing player i's equilibrium strategy, and $\mathbf{G} = (G_i)_{i \in N}$ be the equilibrium. If G_i assigns probability 1 to a single score, it represents a pure strategy.

Lemma 1 (Strategic Equivalence) *A strategy profile is an equilibrium in the contest with additive spillovers if and only if it is an equilibrium in the auxiliary contest.*

Proof. In the auxiliary contest, if the other players use strategies $\mathbf{G}_{-i} = (G_j)_{j \in N \setminus \{i\}}$, player i's expected payoff from choosing s_i is $E[P_i(\mathbf{s}) - K_i(s_i)]$. In the contest with spillovers, if the others players use strategies $\mathbf{G}_{-i}$, player i's expected payoff from choosing s_i becomes $E[P_i(\mathbf{s}) - K_i(s_i)] - E[H_i(\mathbf{s}_{-i})]$, where $E[H_i(\mathbf{s}_{-i})] = \int H_i(\mathbf{s}_{-i}) d\mathbf{G}_{-i}(\mathbf{s}_{-i})$ is independent of his score.[11] The independence is a result of the additive separability. Thus, $\mathbf{G}$ is also an equilibrium

[8] Spillovers through the average or aggregate action are also studied in other competitions, e.g., Acemoglu and Jensen (2013).

[9] This is to ensure the assumption that $r_i \neq r_{m+1}$ for $i \neq m + 1$ is satisfied.

[10] The assumption $\partial C_i(\mathbf{s})/\partial s_i > 0$ requires $h < nc_i$. Otherwise, with $h > nc_i$, it is optimal for player i to choose $s_i = +\infty$.

[11] According to Siegel (2010), each player j's equilibrium strategy G_j is continuous with a finite support. Moreover, H_i is piecewise continuous so is bounded over the supports of $\mathbf{G}_{-i}$. Hence, $E[H_i(\mathbf{s}_{-i})] = \int H_i(\mathbf{s}_{-i}) d\mathbf{G}_{-i}(\mathbf{s}_{-i}) < +\infty$.

in the contest with spillovers. Similarly, the converse is also true, i.e., any equilibrium in the contest with spillovers is also an equilibrium in the auxiliary contest. ■

The result below shows that the original contest with spillovers also has a unique equilibrium, and it is the same one constructed in the auxiliary contest.

Proposition 1 *The all-pay contest with additively spillovers has a unique equilibrium, which is the same as the one that the algorithm of Siegel (2010) constructs for the auxiliary contest.*

Proof. The strategic equivalence (Lemma 1) implies that $\mathbf{G}$ is also an equilibrium in the contest with spillovers. Moreover, suppose there are multiple equilibria in the contest with spillovers. Then, according to the strategic equivalence, there are also multiple equilibria in the auxiliary contest. This is a contradiction because the auxiliary contest has a unique equilibrium. ■

Remark 1 *The equilibrium in the contest with spillovers may* differ *from that in the contest without spillovers. In Example 1, if $h = 0$, there is no spillovers in the original contest, and the scoring costs are $c_1 s_1, ..., c_n s_n$. In contrast, in the auxiliary contest, the scoring costs are $(c_1 - h/n)s_1, ..., (c_n - h/n)s_n$. Because of the different costs, the two contests have different equilibria. Thus, Proposition 1 implies that the original contest with spillovers has a different equilibrium from that in the contest without spillovers.*

According to Proposition 1, we can construct the equilibrium in the contest with spillovers as follows: Given any contest with spillovers, find the corresponding auxiliary contest. Then, apply the algorithm of Siegel (2010) to construct the equilibrium in the auxiliary contest, and this constructed equilibrium is also the equilibrium in the contest with spillovers. Below we illustrate the equilibrium construction for Example 1.

Example 1 (continued) *In the auxiliary contest, player i's cost function is $K_i(s_i) = (c_i - h/n)s_i$. Denote the new marginal cost as $\hat{c}_i \equiv c_i - h/n$. In the equilibrium of this contest characterized by Siegel (2010), player $i = m+2, ..., n$ chooses $s_i = 0$ with probability 1, and player $j = 1, ..., m+1$ mixes over an interval $[s_j^l, s_0^l]$, where $s_m^l = s_{m+1}^l = 0$, $s_0^l = 1/\hat{c}_{m+1}$ and $s_j^l = 1/\hat{c}_{m+1} - \hat{c}_j^{m-j}/(\Pi_{k=j+1}^{m+1}\hat{c}_k)$ for $j = 1, ..., m-1$. Over the interval $[s_j^l, s_{j-1}^l]$, the equilibrium strategy of player $i \in \{j, ..., m+1\}$ is $G_i(s_i) = 1 - (1/\hat{c}_{m+1} - s_i)^{1/(m+1-j)}\beta_{ij}$, where $\beta_{ij} = (\Pi_{k=j}^{m+1}\hat{c}_k^{1/(m+1-j)})/\hat{c}_i$.[12] Proposition 1 implies the above strategy profile is also the unique equilibrium in the original contest with spillovers.*

To illustrate the effect of spillover, we compare the equilibrium in a contest with spillover to that in a contest without. Suppose there is one prize and two players, so $m = 1$ and $n = 2$. If $h = 0$, there is no spillover, and player i's equilibrium strategies are

$$G_1(s; 0) = c_2 s$$
$$G_2(s; 0) = c_1 s + 1 - c_1/c_2$$

[12] The expressions of $G_i(s_i)$ and s_j^l are obtained by substituting $V_i = 1, \gamma_i = \hat{c}_i, \alpha = 1, c(y) = y$ and $a_i = \hat{c}_i$ into (11) and (12) of Siegel (2010).

In contrast, if $h > 0$, there is positive spillover, and the equilibrium strategies become

$$\begin{aligned} G_1(s;h) &= (c_2 - h/2)s \\ G_2(s;h) &= (c_1 - h/2)s + 1 - \frac{c_1 - h/2}{c_2 - h/2} \end{aligned}$$

Note that the spillover affects the players differently. For player 1, who has a lower marginal cost, his strategy with positive spillover first order stochastically dominates that without. For player 2, who has a higher marginal cost, his strategy with spillover intersects with that without.

Although the contest with spillovers has the same equilibrium as the auxiliary contest, the associated equilibrium payoff for a player may differ. This is because positive (negative) spillovers from other players' performance bring additional benefits (costs) to a player. In the auxiliary contest, let $\hat{u}_i$ be player i's equilibrium payoff. Then, the result below characterizes the equilibrium payoffs in the contest with spillovers.

Proposition 2 *In an all-pay contest with additive spillovers, the equilibrium payoff of each player i is $u_i^* = \hat{u}_i - E[H_i(\mathbf{s}_{-i})]$.*

Proof. As in the proof of Lemma 1, given the same equilibrium $\mathbf{G}$, the expected payoff of player i in the contest with spillovers is $E[H_i(\mathbf{s}_{-i})]$ lower than that in the auxiliary contest. Hence, $u_i^* = \hat{u}_i - E[H_i(\mathbf{s}_{-i})]$. ■

Example 1 (continued) *Here we derive the equilibrium payoffs for Example 1. In the auxiliary contest, according to Siegel (2009), the equilibrium payoffs are $\hat{u}_i = 1 - \hat{c}_i/\hat{c}_{m+1} = 1 - (nc_i - h)/(nc_{m+1} - h)$ for players $i = 1, ..., m$, and $\hat{u}_i = 0$ for players $i = m+1, ..., n$. Recall that $H_i(\mathbf{s}_{-i}) = -h(\sum_{j \neq i} s_j)/n$, so Proposition 2 implies that the equilibrium payoffs in the contest with spillover are $u_i^* = \hat{u}_i + h(\sum_{j \neq i} E[s_j])/n$, where $E[s_j]$ is player j' expected scores in the equilibrium constructed above. If h is positive (negative), spillover increases (decreases) the payoff from $\hat{u}_i$ to u_i^*.*

Next, we compare the equilibrium payoffs with spillover to those without. Suppose $m = 1$ and $n = 2$. If $h = 0$, there is no spillover, and the equilibrium payoffs are

$$\begin{aligned} u_1(0) &= 1 - c_1/c_2 \\ u_2(0) &= 0 \end{aligned}$$

If $h > 0$, there is positive spillover, and the equilibrium payoffs become

$$\begin{aligned} u_1(h) &= 1 - \frac{c_1 - h/2}{c_2 - h/2} + \frac{h(c_1 + c_2)}{4(c_2 - h/2)^2} \\ u_2(h) &= \frac{h(c_1 + c_2)}{4(c_2 - h/2)^2} \end{aligned}$$

Notice that the positive spillover increases both players' equilibrium payoffs, but it increases player 1's payoff more than player 2's.

3 Multiplicatively Separable Spillovers

Consider a contest with one prize of value 1 and two players 1 and 2, where we use i to represent one player and j for the other. The model is the same except that the spillover from the other player's score enters the cost in a *multiplicatively separable* way, i.e., $C_i(\mathbf{s}) = K_i(s_i) + L_i(s_i)Q_i(s_j)$ for each i, where $L_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $Q_i : \mathbb{R}_+ \to \mathbb{R}_{++}$ represent asymmetry in costs and spillovers respectively.

We assume that $L_i(0) = 0$ and L_i is continuously differentiable and strictly increasing, and Q_i is continuous. In addition, assume $Q_i(s_j) \geq \underline{q}_i > 0$ for all $s_j \in \mathbb{R}_+$, and there exists $\bar{s} > 0$ such that $L_i(\bar{s})\underline{q}_i > 1$ for all i. As a result, it is never optimal for a player to choose a score above $\bar{s}$, so $Q_i(s_j) \leq \max_{s_j \in [0,\bar{s}]} Q_i(s_j) \equiv \bar{q}_i$. Moreover, player i's cost is zero if $s_i = 0$ and his marginal cost $\partial C_i(\mathbf{s})/\partial s_i = L_i'(s_i)Q_i(s_j)$ is positive because $L_i'(s_i) > 0$ and $Q_i(s_j) > 0$. There is positive spillover if $Q_i(s_j)$ is decreasing in s_j, negative spillover if it is increasing in s_j, and no spillover if it is constant. The above contest is referred as to the two-player contest with multiplicative spillover.

Given any $(q_1, q_2) \in [\underline{q}_1, \bar{q}_1] \times [\underline{q}_2, \bar{q}_2]$, consider an auxiliary contest with the same prize and two players with cost functions $L_i(s_i)q_i$ for $i = 1, 2$. Because q_i is constant, the auxiliary contest has no spillover. As a result, player i's reach is $r_i(q_i) = L_i^{-1}(1/q_i)$ and the threshold is $T(q_1, q_2) = \min(r_1(q_1), r_2(q_2))$. Then, the equilibrium payoff is $u_i(q_1, q_2) = 1 - K_i(T) - L_i(T)q_i$ for player $i = 1, 2$. The equilibrium strategy $G_i(\cdot; q_1, q_2)$ satisfies

$$G_i(s_j; q_1, q_2) - K_i(s_i) - L_j(s_j)q_j = u_j(q_1, q_2) \tag{1}$$

so $G_i(s; q_1, q_2) = u_j(q_1, q_2) + K_i(s_i) + L_j(s_j)q_j$. Given a distribution G_j of s_j, the expectation of $Q_i(s_j)$ is

$$\int Q_i(s)dG_j(s) \equiv E[Q_i(s_j)|G_j] \tag{2}$$

where the integration is over the support of G_j. Then, we can define a mapping $\phi : [\underline{q}_1, \bar{q}_1] \times [\underline{q}_2, \bar{q}_2] \to [\underline{q}_1, \bar{q}_1] \times [\underline{q}_2, \bar{q}_2]$ such that

$$\phi(q_1, q_2) = (E[Q_1(s_2)|G_2(\cdot; q_1, q_2)], E[Q_2(s_1)|G_1(\cdot; q_1, q_2)])$$

For $i = 1, 2$, let $\hat{r}_i$ be the unique solution of[13]

$$\int_0^{\hat{r}_i} Q_i(s)dL_i(s) = 1 \tag{3}$$

Without loss of generality, rename the players so that $\hat{r}_1 \geq \hat{r}_2$. Using the mapping, the following result links the original contest to the auxiliary contest.

[13]To see why there is a unique solution, notice that if $\hat{r}_i = 0$, the left hand side (LHS) of (3) is 0, which is smaller than 1. In contrast, if $\hat{r}_i = \bar{s}$, the LHS is larger than $\underline{q}_i L_i(\bar{s}) > 1$. Therefore, (3) has at least one solution. Notice that the LHS of the above equation is strictly increasing in $\hat{r}_i$, so (3) has a unique solution.

Lemma 2 *If* $\{G_1, G_2\}$ *is an equilibrium of the original contest,* $(E[Q_1(s_2)|G_2], E[Q_2(s_1)|G_2])$ *is a fixed point of* ϕ*. If* (q_1, q_2) *is a fixed point of* ϕ*, then* $\{G_1(\cdot; q_1, q_2), G_2(\cdot; q_1, q_2)\}$ *is an equilibrium of the original contest.*

Proof. Suppose $\{G_1, G_2\}$ is an equilibrium of the original contest. Then, $\{G_1, G_2\}$ is the unique equilibrium in the auxiliary contest with $q_1 = E[Q_1(s_2)|G_2]$ and $q_2 = E[Q_2(s_1)|G_1]$. Hence, the definition of ϕ implies $\phi(q_1, q_2) = (E[Q_1(s_2)|G_2], E[Q_2(s_1)|G_1])$, so $(E[Q_1(s_2)|G_2], E[Q_2(s_1)|G_1])$ is a fixed point of ϕ.

Suppose (q_1, q_2) is a fixed point of ϕ. In the auxiliary contest, $\{G_1(\cdot; q_1, q_2), G_2(\cdot; q_1, q_2)\}$ is an equilibrium, which means, given other's strategy $G_j(\cdot; q_1, q_2)$, player i does not deviate from $G_i(\cdot; q_1, q_2)$. Then, given the other's strategy $G_j(\cdot; q_1, q_2)$, player i's payoff from choosing s_i in the original contest is $G_j(s_i, q_1, q_2) - L_i(s_i)E[Q_i(s_j)|G_j(\cdot; q_1, q_2)]$, which equals to $G_j(s_i, q_1, q_2) - L_i(s_i)q_i$, his payoff from choosing s_i in the auxiliary contest because $E[Q_i(s_j)|G_j(\cdot; q_1, q_2)] = q_i$ due to the definition of fixed point. Therefore, player i does not deviate from $G_i(\cdot; q_1, q_2)$ either, which means $\{G_1(\cdot; q_1, q_2), G_2(\cdot; q_1, q_2)\}$ is an equilibrium in the original contest. ■

As a result, if we find a unique fixed point of ϕ, we also find a unique equilibrium in the original contest. We first introduce some notation, which is used to characterize the unique equilibrium. The following result characterizes a unique equilibrium.

Proposition 3 *The two-player contest with multiplicative spillover has a unique equilibrium. In the equilibrium, the payoff is* $u_1^* = 1 - L_1(\hat{r}_2)q_1^*$ *for player* 1 *and* $u_2^* = 0$ *for player* 2*, the strategies are* $G_i(s_i) = u_j^* + L_j(s_i)q_j^*$ *for* $s_i \in [0, \hat{r}_2]$*, where*

$$q_1^* = \frac{Q_1(0)}{1 + L_1(\hat{r}_2)Q_1(0) - \int_0^{\hat{r}_2} Q_1(s)dL_1(s)} \tag{4}$$

$$q_2^* = \frac{1}{L_2(\hat{r}_2)} \tag{5}$$

Proof. We first verify the above equilibrium. In the auxiliary contest with q_1^*, q_2^*, the equilibrium strategies are G_i described in the proposition. Then,

$$\begin{aligned} E[Q_1(s_2)|G_2] &= \int_0^{\hat{r}_2} Q_1(s_2)dG_2(s_2) \\ &= Q_1(0)G_2(0) + q_1^* \int_0^{\hat{r}_2} Q_1(s_2)dL_1(s_2) \\ &= Q_1(0)(1 - L_1(\hat{r}_2)q_1^*) + q_1^* \int_0^{\hat{r}_2} Q_1(s_2)dL_1(s_2) \qquad (6) \\ &= q_1^* \end{aligned}$$

where the second equality is from the atom of G_2 at 0, the third from $G_2(0) = u_1 = 1 - L_1(\hat{r}_2)q_1^*$, and the last from $Q_1(0) = q_1^*[1 + L_1(\hat{r}_2)Q_1(0) - \int_0^{\hat{r}_2} Q_1(s)dL_1(s)]$ due to (4). In addition,

$$E[Q_2(s_1)|G_1] = q_2^* \int_0^{\hat{r}_2} Q_2(s_1)dL_2(s_1) = q_2^*$$

where the second equality is from the definition of $\hat{r}_2$. Hence, (q_1^*, q_2^*) is a fixed point of ϕ, so, according to Lemma 2, the above strategies constitute an equilibrium in the original contest.

To prove the equilibrium uniqueness in the original contest, it is sufficient to show that ϕ has a unique fixed point. Suppose (q_1, q_2) is a fixed point of ϕ. First, we show that $r_1(q_1) \geq r_2(q_2)$ in the auxiliary contest with q_1, q_2. To see why, suppose otherwise that $r_1(q_1) < r_2(q_2)$. Then, in the auxiliary contest, the threshold is $T = \min\{r_1(q_1), r_2(q_2)\} = r_1(q_1)$ and the equilibrium payoffs are $u_1(q_1, q_2) = 0$ and $u_2(q_1, q_2) = 1 - L_2(r_1(q_1))q_2$. Moreover, (1) implies the equilibrium strategies are

$$\begin{aligned} G_1(s; q_1, q_2) &= L_2(s)q_2 + 1 - L_2(r_1(q_1))q_2 \\ G_2(s; q_1, q_2) &= L_1(s)q_1 \end{aligned}$$

Therefore,

$$\begin{aligned} E[Q_1(s_2)|G_2(\cdot; q_1, q_2)] &= q_1 \int_0^{r_1(q_1)} Q_1(s)dL_1(s) \\ E[Q_2(s_1)|G_1(\cdot; q_1, q_2)] &= Q_2(0)[1 - L_2(r_1(q_1))q_2] + q_2 \int_0^{r_1(q_1)} Q_2(s_2)dL_2(s_2) \end{aligned}$$

where the second equation is from the same calculation to obtain (6). Hence, $(q_1, q_2) = \phi(q_1, q_2)$ is equivalent to

$$q_1 = q_1 \int_0^{r_1(q_1)} Q_1(s)dL_1(s) \tag{7}$$

$$q_2 = Q_2(0)[1 - L_2(r_1(q_1))q_2] + q_2 \int_0^{r_1(q_1)} Q_2(s_2)dL_2(s_2) \tag{8}$$

Equation (8) implies

$$q_2 = \frac{Q_2(0)}{1 + L_2(r_1(q_1))Q_2(0) - \int_0^{r_1(q_1)} Q_2(s)dL_2(s)}$$

Substituting the above q_2 into $L_2(r_2(q_2))q_2 = 1$, which is required by the definition of $r_2(q_2)$, and rearranging terms, we obtain

$$[L_2(r_2(q_2)) - L_2(r_1(q_1))]Q_2(0) = 1 - \int_0^{r_1(q_1)} Q_2(s)dL_2(s) \tag{9}$$

Comparing (7) to (3), we obtain $r_1(q_1) = \hat{r}_1$. Recall that $\hat{r}_1 \geq \hat{r}_2$, so the right hand side of (9) is smaller than $1 - \int_0^{\hat{r}_2} Q_2(s)dL_2(s)$, which is zero by the definition of $\hat{r}_2$. Moreover, we assume above that $r_1(q_1) < r_2(q_2)$, so the left hand side of (9) is positive. Hence, (9) is violated, which is a contradiction.

As a result, we must have $r_1(q_1) \geq r_2(q_2)$ if (q_1, q_2) is a fixed point. Then, in the auxiliary contest, the threshold is $T = r_2(q_2)$. Repeating the same analysis, we obtain the analogues of

(7) and (8):

$$q_2 = q_2 \int_0^{r_2(q_2)} Q_2(s) dL_2(s)$$
$$q_1 = Q_1(0)[1 - L_1(r_2(q_2))q_1] + q_1 \int_0^{r_2(q_2)} Q_1(s_1) dL_1(s_1)$$

whose unique solution is (q_1^*, q_2^*) described in (4) and (5). Therefore, there cannot be other fixed points of ϕ. ■

Example 2 *For $i = 1, 2$, suppose $C_i(s_1, s_2) = c_i s_i \bar{s}$. Therefore, $C_i(s_1, s_2) = c_i s_i^2/2 + c_i s_i s_j/2 = K_i(s_i) + L_i(s_i)Q_i(s_j)$, where $K_i(s_i) = c_i s_i^2/2$, $L_i(s_i) = c_i s_i$ and $Q_i(s_j) = s_j/2$ with $c_i, \beta_i > 0$. The other player's score s_j has negative spillover on player i if $\beta_i > 1$, positive spillover if $\beta_i < 1$ and no spillover if $\beta_i = 1$.*

Three or More Players With multiplicative spillovers, if there are $m \geq 1$ prizes of value 1 and $n \geq 3$ players, we can no longer solve the equilibrium payoffs and strategies explicitly. Given $q_1, ..., q_n$, we can still introduce an auxiliary contest similarly. However, two difficulties arise: First, unlike the two-player case, some players may choose zero performance with certainty, and it is difficult to determine who they are.[14] Second, if we can determine the actively competing players, the equilibrium strategies in the auxiliary contest may be very complicated. For example, Siegel (2010) shows gaps in the support of the mixed strategies. This makes it difficult to derive the explicit expression of ϕ as in (7) and (8), so it is difficult to solve the fixed points of ϕ.

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[14] For some values of $q_1, ..., q_n$, there may be multiple players's reaches equal to the threshold, which violates Assumption B3 of Siegel (2010). Therefore, we cannot pin down the equilibrium payoffs and strategies in the auxiliary contest.

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